The Timing of Annuitization: Investment Dominance and Mortality Risk

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Abstract

TITLE: **The Timing of Annuitization: Investment Dominance and Mortality Risk**

We use preference-free dominance arguments to develop a framework for locating the optimal age (time) at which to purchase an irreversible life annuity. Then, using annuity-pricing mortality tables and the characteristics of real-world annuity payouts, we show that annuitization prior to age 65-70 is dominated by self-annuitization even in the absence of any bequest motives. For retirees who are willing to accept some risk in exchange for retaining the benefits of liquidity and bequest, the optimal age can be even higher. Aside from the normative implications within the context of proposals to reform Social Security, these results should help shed light on the annuity puzzle by focusing researcher's attention on the *specific ages* for which a puzzle can actually be said to exist.

1 The Main Claim

There are two stylized facts about the market for fixed and variable-payout annuities that impede Yaari's (1965) full annuitization results, as well as the recent extension by Davidoff, Brown, and Diamond (2005). These facts, which have not received much attention in the insurance and economics literature, shed light on the so-called annuity puzzle and the low rates of voluntary annuitization described at length by Modigliani (1986) and others. Indeed, according to a recent survey by the U.S.-based DSG company, less than $500 million in premium was invested in fewer than 4,000 variable-payout annuity contracts during the year 2003. This number contrasts with more than $120 billion of premium that flowed into deferred variable annuities during the same period. The annuitization rate on VA contracts – which only contain an option to annuitize and not a requirement to do so – was less than 2% according to a survey by Sondergeld (1997). This is the essence of the annuity puzzle.

However, one fact that is critical to understanding the payout (a.k.a. income) annuity market is that despite the power of the law of large numbers to diversify individual and idiosyncratic mortality risk, all insurance companies that offer payout annuities impose an ongoing asset-based mortality risk fee to compensate for taking aggregate longevity risk. Note that since insurance companies guarantee lifetime income, they are at risk of loss if the population as a whole lives longer than anticipated. While some economists – for example Friedman and Warshawsky (1990) – might classify any additional fee under the umbrella of transaction costs, this particular fee is an inseparable component of aggregate mortality risk and creates a unique impediment to annuitization, as we shall demonstrate.

The reason for this unavoidable, yet critical, asset-based mortality fee is that there are two distinct kinds of mortality risk that an insurance company is exposed to when selling mortality-contingent claims. The first risk is that a particular insured life lasts longer than average. This risk can be diversified by selling more claims and using the law of large numbers to drive the per-policy risk to zero. The other kind of mortality risk is the risk of underestimating population mortality rates – for example, the risk of a biomedical discovery that uniformly reduces population mortality – and this risk cannot be diversified by selling more policies. In fact, a larger portfolio of insurance policies will only increase this risk. Few authors have focused on the impact of this risk and the inevitable fees charged by insurance companies to cover this risk, on the optimal timing and demand for annuities.

A second often-neglected fact regarding fixed and variable-payout annuities is that the menu of investment choices available under the annuity wrapper is dominated by – or at the very least, inferior to in a mean/variance sense – products available outside the annuity structure. Thus, while most U.S.-based investors can select over 5,000 mutual funds in addition to many thousands of individual securities traded on major stock exchanges around the world for their personal portfolio, the set of options available within variable-payout annuities is quite limited. This restriction, too, can be placed under the umbrella of known economic frictions, namely, market incompleteness. However, we believe that we are the first to study the impact of currently available annuity products on the optimal timing of annuitization.

TABLE #1 PLACED HERE

Table #1 provides (what we believe to be) a comprehensive listing of all companies offering variable-payout annuities in the U.S., together with their current menu of investment offerings. For example, TIAA-CREF charges 0.40% (40 basis points) of assets each year to cover mortality risk. This ongoing fee is above any investment management fee – which can range from an additional 5 to 50 basis points – that would be imposed depending on the portfolio of sub-accounts selected. Yet, while they offer the cheapest payout annuity, TIAA-CREF only allows 8 funds to choose from. These funds run the gamut from a generic equity-index fund to a real-return bond fund, as well as a real estate fund. And, while TIAA-CREF (or any other company for that matter) might claim that 8 funds are enough to span the risk and return spectrum, this view is clearly not shared by all.[1] Other companies charge more for taking on mortality risk but offer a slightly expanded menu of investment sub-accounts. We do not delve into the efficiency of the sub-accounts or whether these accounts span the entire risk-return trade-off in a Markowitz sense. Our only relevant comment is that the entire universe of investable assets is <u>not</u> available within these payout annuities. It is currently impossible – due to both securities and insurance regulation – to acquire individual stocks, residential real-estate, commodities or basic savings bonds within a payout annuity.

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[1] A number of univerisites and colleges that are part of the TIAA-CREF pension system have been petitioning the relevant authorities to "allow" more choice in their annuity sub-accounts.

In sum, our main claim is that the absence of a complete menu of investment products, in conjunction with a perpetual fee to cover the insurance company's aggregate mortality risk, leads to a situation under which a self-annuitization strategy, which we will precisely describe in a moment, can dominate the payout annuity and lead to the optimality of delayed annuitization. Formally, the definition of dominance under which we operate is as follows: Denote the random income payoffs from the strategy to annuitize later by A and from the strategy to annuitize now by B. We say that random payoff A *dominates* random payoff B, if for every possible state of nature (a.k.a. possible investment return), the payoff under strategy A is at least as great as the payoff under B. If that is the case, then it is worthwhile to delay.

Thus, we do not resort to any preference-based (that is, utility-based) models of consumption and bequest motives. In fact, given these facts, we find little justification for annuitizing any additional wealth prior to ages 65 - 70. Interestingly enough, these numbers coincide with the proposal by Feldstein (2005), under which *individual accounts* would be forcefully annuitized at age 67. The only justification for premature annuitization is that the mortality and hazard rates embedded within current annuities will decline over time and, hence, increase the future cost of longevity insurance[2]. However, we believe there are numerous dynamic inconsistencies with this claim because it implicitly assumes that insurance actuaries are myopic and do not incorporate all available information about expected future hazard rates at the time of sale. Also, even if one is to assume that current market-based hazard rates are cheap, our model forces consumers to acknowledge that annuitizing early is an active bet against the mortality forward-curve embedded within the annuity quotes, as opposed to an attempt to lock-in mortality credits.

We now proceed by developing a simple model to illustrate the impact of limited menus and asset-based risk fees on the optimal timing of annuitization. We develop a more sophisticated model using continuous-time stochastic calculus after the intuition has been laid out with the simple model.

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[2] See for example Brugiavini (1993) who develops a model along these lines.

2 The Simple Model

A retiree aged x with w liquid dollars of investable wealth can purchase a *variable*-payout (a.k.a. income) annuity that entitles the annuitant to w/a_x annuity units, where a_x is the actuarial annuity factor defined by

$$a_x = \sum_{i=1}^{\infty} \frac{{}_ip_x}{(1+H)^i}, \tag{1}$$

in which ${}_ip_x$ is the conditional survival probability that a person aged x survives an additional i years, and H is the assumed (a.k.a. anticipated) interest rate (AIR) that determines the desired rate at which payments increase over time. The assumed interest rate can be selected by the annuitant within allowable ranges, typically between 2% and 8%. For example, if $a_x = 15$, then $w = 100{,}000$ dollars will buy $100{,}000/15 = 6666.667$ annuity units, where each unit has an initial value of \$1.

By simple algebra, if the retiree delays annuitization by one year to age $x+1$, the annuity factor – which is the cost of acquiring the same payout at that time – will be

$$a_{x+1} = \sum_{i=1}^{\infty} \frac{{}_ip_{x+1}}{(1+H)^i} = \left(\frac{1+H}{1-q_x}\right) \sum_{i=1}^{\infty} \frac{{}_{i+1}p_x}{(1+H)^{i+1}} = \left(\frac{1+H}{1-q_x}\right) a_x - 1, \tag{2}$$

since by definition: $({}_ip_{x+1})({}_1p_x) = {}_{i+1}p_x$. Here, $q_x = 1 - {}_1p_x$ denotes the one-year probability of dying for a person aged x. The only economic assumption implicit in equation (2) is that the q_{80}, for example – which is the conditional probability of death at age 80 – built into the annuity pricing equation is the same number regardless of whether the annuity is purchased at age 65, 70 or beyond. We will use this identity in a moment.

Note that if the retiree annuitizes at age x, then at the end of the first year, each annuity unit will generate a variable income of

$$\frac{1 + \mathbf{R}^a - M}{1+H}, \tag{3}$$

in which $\mathbf{R}^a$ denotes the random return of the underlying portfolio supporting the payout annuity, H denotes the AIR and M denotes the above-mentioned fee that is charged by the insurer to cover aggregate mortality risk. For example, if the sub-account portfolio earns a net $\mathbf{R}^a = 10\%$ after all investment management fees have been deducted, but the insurance company charges $M = 0.5\%$ for aggregate mortality risk and the annuitant selected an

$H = 5\%$ AIR, then the (end of the) first year's income flow will be $(1 + 0.1 - 0.005)/(1 + 0.05)$ =1.042857 dollars per annuity unit. Thus, if the retiree acquired 100,000/15 = 6666.667 units, the (end of the) first year's income flow will be (6666.667)(1.042857) = 6952.38 dollars. In algebraic terms, the annuitant will receive $\mathbf{c}_1 = (w/a_x)(1 + \mathbf{R}^a - M)/(1 + H)$ dollars at the end of the first year, which is random at time zero.

At the end of the second retirement year the annuitant will receive $\mathbf{c}_2 = \mathbf{c}_1(1 + \mathbf{R}^a - M)/(1+H)$ dollars, where $\mathbf{R}^a$ is separate realization of the same random variable, and so on for future years. Intuitively, selecting a higher AIR value of H leads to a lower annuity factor in equation (1), which will result in a greater number of acquired annuity units. However, the larger denominator in equation (3) implies that payments will increase at a lower rate, since the return must overcome the hurdle rate of H before payments increase. Of course, on an actuarial present value basis, all values of AIR lead to equivalent annuity entitlements and there is no natural advantage to selecting one over the other. For an economic perspective on the mechanics of variable payout annuities, please see the paper by Feldstein and Ranguelova (2001) within the context of proposals for Social Security reform.

We now present the core of our dominance argument. Assume that at age x, the retiree decides to (i) invest the entire w directly in an investment portfolio earning a random $\mathbf{R}^w$, which might differ from $\mathbf{R}^a$, (ii) withdraw the random $\mathbf{c}_1$ at the end of the year, and then (iii) annuitize instead at age $x+1$. The retiree's strategy of delayed annuitization will dominate the variable-payout annuity if

$$\frac{w(1+\mathbf{R}^w) - \left(\frac{w}{a_x}\right)\left(\frac{1+\mathbf{R}^a-M}{1+H}\right)}{a_{x+1}} \geq \frac{w}{a_x}\left(\frac{1+\mathbf{R}^a-M}{1+H}\right), \tag{4}$$

in which we mean that the inequality holds for all states of the world. This is a rather strong view of dominance; it is also referred to as *monotonicity.*[3] If the left-hand side of equation (4) is greater than the right-hand side for all possible outcomes of $\mathbf{R}^a$ and $\mathbf{R}^w$, then it makes no sense to annuitize at age x.

Note that the numerator of the left-hand side of equation (4), represents what remains of liquid wealth after withdrawing the self-annuitized amount. The denominator of the left-hand side of equation (4) converts this amount into annuity units at the same AIR H, so we

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[3] One could replace the inequality in (4) with weaker notions of dominance, including first- and second-stochastic dominance (Hadar and Russell, 1969; Hanoch and Levy, 1969; Rothschild and Stiglitz, 1970).

are consistently comparing the same income flow going forward.

Inequality (4) implies that the self-annuitization strategy will dominate the immediate annuitization strategy at age x, as long as

$$(1+\mathbf{R}^w) - \left(\frac{1}{a_x}\right)\left(\frac{1+\mathbf{R}^a - M}{1+H}\right) \geq \left(\left(\frac{1+H}{1-q_x}\right)a_x - 1\right)\frac{1}{a_x}\left(\frac{1+\mathbf{R}^a - M}{1+H}\right), \quad (5)$$

where we make use of the relationship between a_{x+1} and a_x from equation (2). Inequality (5), then, leads to the main condition under which delaying dominates immediately annuitizing:

$$M \geq q_x + (\mathbf{R}^a - \mathbf{R}^w) + q_x\mathbf{R}^w. \quad (6)$$

If the annualized mortality risk charge M is greater than the one-year probability of dying plus a function of the random return, then immediate annuitization is dominated by waiting for one year. In other words, even with zero utility of bequest – or a Yaari (1965) world – it makes to sense to annuitize at age x, since alternative assets can do better. Of course, as x and the value of $q_x$increases, it becomes less likely that M will exceed the relevant threshold and so annuitization is optimal.

For example, assume that $\mathbf{R}^a \equiv \mathbf{R}^w$, which implies that the same exact portfolio is available within the annuity structure and outside the annuity. In that case, the condition for dominance is that $M \geq q_x(1+\mathbf{R}^a)$. Thus, if the random variable $\mathbf{R}^a$ is bounded above by some constant $\widehat{R^a}$, and if $M \geq q_x(1+\widehat{R^a})$, then the strategy of immediate annuitization is dominated by waiting for one year. A reasonable number for the upper bound $\widehat{R^a}$ might be 50% – since a well diversified portfolio will not likely earn more than 50% in a given year – although its precise magnitude is less important than the fact that some upper bound generally exists. Later, when we present the continuous-time multiperiod model under which portfolio rebalancing is available continuously, we will show that dominance can exist even without imposing the upper bound.

TABLE #2 PLACED HERE

To illustrate the practical implications, Table #2 provides a sample of mortality rates at various ages. These are not necessarily population mortality rates – such as those compiled by the Social Security Administration – but are actual annuitant mortality rates that are used in the pricing of immediate annuities and are relevant to this exercise. Our dominance argument is not predicated on retirees (consumers) knowing their subjective mortality rates.

According to Table #2, at age 65, insurance companies assume that female annuitant will die within the next year with probability $q_{65} = 0.00625$. This implies that if the upper bound on the sub-account investment returns is 50%, then as long as the mortality risk fee $M \geq 0.00625(1.5) = 0.009\,375$, which is approximately 94 basis points, the strategy of immediate annuitization is dominated. Note that 6 out of the 10 annuities listed in Table #1 have mortality risk fees that are greater than 94 basis points. In fact, the four companies that do charge a lower mortality risk fee explicitly state in their documentation that they retain the option of increasing these fees at their discretion.

As an aside, equations (4) to (6) also cover the case for which no variable-payout annuities are available so that the market consists only of fixed-payout annuities.[4] In this case, there is no mortality risk charge, so $M = 0$. The assumed interest rate H is replaced by a pricing rate R and the random variable $\mathbf{R}^a$ is by definition equal to the constant R, so that the income profile is flat. In this case, the dominance delay condition in equation (6) can be written as:

$$\mathbf{R}^w \geq \frac{1+R}{1-q_x} - 1. \tag{7}$$

As long as (one plus) the non-annuitized portfolio return is greater than (one plus) the annuity pricing rate divided by the survival probability, immediate annuitization is dominated by delaying. This idea, together with various simulations of these probabilities, was pursued in Milevsky (1998).

In sum, by focusing-in on the actual details of realistic payout annuity markets, we have derived simple conditions under which it does and does not make sense to annuitize.

3 A More Complicated Model

We now generalize the dominance condition to a multi-period continuous-time model in which portfolios can be rebalanced continuously. As before, the retiree starts with $W_0 = w$ at retirement. We start with a portfolio/index underlying the annuity sub-account that follows geometric Brownian motion:

$$\mathbf{S}_t = e^{(\mu - 0.5\sigma^2)t + \sigma \mathbf{B}_t}, \qquad \mathbf{S}_0 = 1, \tag{8}$$

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[4] This is the situation currently faced in many jurisdictions (for example, until recently, Canada).

in which μ represents the investment drift rate, net of any investment management fees, and σ represents the diffusion rate (a.k.a. volatility) of the annuity sub-account. This model is the “working horse" of option pricing and portfolio choice, as originally developed by Merton (1971). A more complicated model that includes jumps and/or stochastic volatility would not contribute much to the underlying economics of the problem.

If the retiree buys a variable-payout annuity at age x that is linked to the above portfolio/index, she will receive a continuous stream of income for life which obeys the stochastic process:

$$\mathbf{c}_t = \frac{w}{\overline{a}_x} e^{(\mu - 0.5\sigma^2 - m - h)t + \sigma \mathbf{B}_t}, \tag{9}$$

in which $\overline{a}_x$ is the continuous-time analogue of the annuity factor in equation (1), h is the assumed interest rate, and m is the asset-based mortality risk fee – all using continuous compounding. Intuitively, the income flow will increase in proportion to $(\mu - 0.5\sigma^2)$ which is the growth rate (or geometric mean) of the underlying portfolio. The income flow will increase as a slower rate the greater the assumed interest rate (AIR) h, and/or the greater the mortality risk fee m. The uncertainty in the financial market is captured by the Brownian motion $\mathbf{B}_t$, which is a zero-mean and t-variance random variable. The expected income flow at any time T, can be computed by taking expectations of the random component $E[e^{\sigma \mathbf{B}_T}]$, and is equal to $(w/\overline{a}_x)e^{(\mu - m - h)T}$. The income from a variable payout annuity is expected to increase over time provided that the arithmetic mean return μ, is greater than the mortality risk fee plus the AIR.

Similar to the arguments made in the previous section, if the retiree decides to delay annuitization for T years and self-annuitize in the meantime, the marketable portfolio will satisfy the following diffusion process:

$$d\mathbf{W}_t = (\mu + l)\mathbf{W}_t dt - \mathbf{c}_t dt + \sigma \mathbf{W}_t d\mathbf{B}_t, \qquad \mathbf{W}_0 = w, \tag{10}$$

where the new parameter l is what we call the *dominating portfolio spread* (DPS).

The DPS represents the incremental return above μ that may be available from non-annuitized investments outside the annuity sub-accounts, yet subject to the same level of risk σ. One can think of this statement within a classical Markowitz framework, in which an inefficient portfolio pair (μ, σ) can be improved by rebalancing the asset weights and maintaining the same level of risk, but increasing the portfolio return to $(\mu + l, \sigma)$. Equation

(10) enables one to answer the question: “How much more can I expect to earn from an efficient portfolio outside the annuity structure by taking on the same level of risk?"

Thus, for the delay strategy to dominate the immediate annuitization strategy, the following relationship must hold for all outcomes of $\mathbf{W}_T$ and $\mathbf{B}_T$:

$$\frac{\mathbf{W}_T}{\bar{a}_{x+T}} \geq \mathbf{c}_T = \frac{w}{\bar{a}_x} e^{(\mu - 0.5\sigma^2 - m - h)T + \sigma \mathbf{B}_T}. \tag{11}$$

The left-hand side represents investable wealth $\mathbf{W}_T$ at time T divided by the relevant annuity factor at time T. The right-hand side captures the consumption flow at time T, assuming the initial wealth w was annuitized at time zero. Inequality (11) is the continuous-time analogue of inequality (4). The dominance condition can, thus, be written as

$$\mathbf{W}_T e^{-(\mu - 0.5\sigma^2 - m - h)T - \sigma \mathbf{B}_T} \geq w \frac{\bar{a}_{x+T}}{\bar{a}_x}, \tag{12}$$

where – at least initially – it seems that the left-hand side of inequality (12) is a random variable, while the right hand side is not.

Now, based on elementary methods from the theory of stochastic differential equations – see for example Karatzas and Shreve (1998, Chapter 5.6, Section C) – one can solve equation (10) for $\mathbf{W}_T$. Indeed,

$$\mathbf{W}_T = e^{(\mu + l - 0.5\sigma^2)T + \sigma \mathbf{B}_T} \left(w - \int_0^T \mathbf{c}_s e^{-(\mu + l - 0.5\sigma^2)s - \sigma \mathbf{B}_s} ds \right). \tag{13}$$

Recall from the definition of $\mathbf{c}_s$ in equation (9) that the integrand portion of equation (13) can be written as

$$\mathbf{c}_s e^{-(\mu + l - 0.5\sigma^2)s - \sigma \mathbf{B}_s} = \frac{w}{\bar{a}_x} e^{(\mu - 0.5\sigma^2 - m - h)s + \sigma \mathbf{B}_s} e^{-(\mu + l - 0.5\sigma^2)s - \sigma \mathbf{B}_s} = \frac{w}{\bar{a}_x} e^{-(m+h+l)s}. \tag{14}$$

Thus, we have succeeded in eliminating any stochastic component from the integral within equation (13).

At any time T, the value of $\mathbf{W}_T$ is a geometric Brownian motion, albeit multiplied by a time-dependent function. By integrating the exponential term $e^{-(m+h+l)s}$ from zero to T, equation (13) – which is the wealth of the self-annuitizer – can be written as

$$\mathbf{W}_T = w e^{(\mu + l - 0.5\sigma^2)T + \sigma \mathbf{B}_T} \left(1 - \frac{1 - e^{-(m+h+l)T}}{(m+h+l)\bar{a}_x} \right). \tag{15}$$

The left-hand side of equation (12), which initially appeared to be random, can now be simplified and written as

$$\mathbf{W}_T e^{-(\mu-0.5\sigma^2-m-h)T-\sigma \mathbf{B}_T} = w e^{(m+h+l)T}\left(\frac{(m+h+l)\overline{a}_x - 1 + e^{-(m+h+l)T}}{(m+h+l)\overline{a}_x}\right), \tag{16}$$

which does *not* contain the random element $\mathbf{B}_T$. Substituting equation (16) into the dominance condition presented in inequality (12) leaves us with

$$e^{(m+h+l)T}\left(\frac{(m+h+l)\overline{a}_x - 1 + e^{-(m+h+l)T}}{m+h+l}\right) \geq \overline{a}_{x+T}. \tag{17}$$

In sum, regardless of the random outcome of $\mathbf{B}_T$ during the period $[0,T]$, the strategy of delaying annuitization will dominate immediate annuitization if inequality (17) is satisfied, which is a deterministic relationship among the parameters of the model.

This condition can be written more simply as

$$e^{\delta T}a_0 - a_1 - \frac{e^{\delta T}-1}{\delta} \geq 0, \tag{18}$$

in which $a_1 = \overline{a}_{x+T}$ and $a_0 = \overline{a}_x$ denote the two annuity factors, and $\delta = m+h+l$. Thus, for any given delay period T, and for any value of l, m and h, inequality (18) dictates whether immediate annuitization is dominated by delaying.

There is a unique value of the DPS l that solves inequality (18) with equality. Indeed, by multiplying both sides of equation (18) by $e^{-\delta T}$, and by isolating δ on one side, we get that

$$a_0 = a_1 e^{-\delta T} + \frac{1-e^{-\delta T}}{\delta}. \tag{19}$$

The right-hand side (RHS) decreases with respect to delta. Note that for $\delta = 0$, the RHS equals $a_1 + T$, which is greater than a_0. As δ goes to infinity, the RHS goes to zero. Therefore, there is a unique value of δ that solves the equation. Because m and h are given, it follows that the corresponding value of l is unique.

Inequality (18) has an economic interpretation as well. Because the randomness in the consumption stream exactly follows the randomness in the wealth and annuity processes, the dominance condition reduces to a deterministic inequality. The problem scales, so assume that the retiree has wealth a_0, the amount required to purchase a continuous annuity that pays at the rate of \$1 for the rest of her life if annuities were priced using the interest rate h. Instead of purchasing this annuity, suppose that the retiree decides to invest the money

a_0 and self-annuitize. The first term on the left-hand side of inequality (18) is the value of the retiree's investment at time T if that money were to earn the rate of $\delta = m + h + l$. The second term is the cost of the annuity at time T, while the third term is the value of the self-annuitized stream of \$1 during $[0, T]$ at time T, again assuming the interest rate δ. Therefore, the left-hand side of inequality (18) is the amount of money remaining at time T from this set of transactions. If this remnant is positive, then the retiree is better off waiting to annuitize at time T rather than annuitizing immediately. Note that the "interest rate" δ is independent of μ and σ.

4 Numerical Examples

The following table, Table #3, illustrates a menu of initial annual payouts available on a variable-payout annuity at ages 65 and 70 under a variety of assumed interest rates.

TABLE #3 PLACED HERE

For the purposes of verification, the numbers in Table #3 were obtained from an actuarial model that assumes survival probabilities obey the following law of mortality:

$$\ln({}_tp_x) = e^{\left(\frac{x-m}{b}\right)} \left(1 - e^{t/b}\right), \tag{20}$$

in which ${}_tp_x$ is the conditional survival probability, and m, b are two free parameters. This is known as the Gompertz law of mortality, which is widely used in actuarial and economic studies. We further assume that $m = 90$ and $b = 9.5$, which leads to an expected remaining lifetime of 21.69 years at age $x = 65$. The one-year conditional probability of survival at age 65 closely matches the discrete (unisex) number implied by Table #2.

Thus, for example, under an AIR of $h = 3\%$, the annualized annuity factor at age 65 is $a_0 = 100{,}000/6552.65 = 15.261$, and at age 70, it is $a_1 = 100{,}000/7639.42 = 13.09$. Regardless of whether the annuity is purchased at age 65 or at age 70, payments will increase each year based on the random performance of the underlying sub-account above the AIR of 3%, minus a mortality fee of 80 basis points. The dynamics of subsequent payments are specified by equation (9).

If the 65-year-old decides to delay annuitization until age 70, then in order for this decision to dominate annuitizing immediately, she must earn at least the dominating portfolio spread (DPS) l, which solves the break-even condition underlying inequality (18), namely

$$a_1 - e^{(l+0.008+0.03)5}\left(a_0 - \frac{1}{l+0.008+0.03}\right) - \frac{1}{l+0.008+0.03} = 0, \tag{21}$$

If we substitute $a_1 = 13.09$ and $a_0 = 15.261$, and then solve for l, we get 18.1 basis points as the spread required for dominance. The retiree would have to earn 18.1 basis points per year (during the five deferral years) above what the investment sub-accounts earn to be able to purchase the same lifetime annuity stream in five years. According to Table #3, when the AIR is $h = 5\%$, the equivalent value is 17.8 basis points, and when $h = 7\%$ the DPS is 17.4 basis points. The intuition as to why l is larger for smaller values of h is that for smaller values of h, the variable payments will grow at a faster rate and, hence, the self-annuitizer must work harder to keep up.

Note, also, that if instead of using $m = 80$ basis points, we were to use 100 basis points for the mortality fee – as do most of the companies listed in Table #1 – the dominating portfolio spread (DPS) is decreased by 20 basis points for all AIR values. This leads to negative value of l, which means that even if the self-annuitizer invests in an inferior portfolio that loses 2-3 basis points relative to the annuity sub-accounts, she will be better off by waiting.

To sum up, we have derived a simple preference-free rule for determining the optimal time to annuitize based on dominance arguments. For any set of initial variable annuity payments, such as those listed in Table #3, and for a given mortality risk charge, we have computed the (break-even) dominating portfolio spread that leads to a deferral.

5 Conclusion

The extant economics literature is markedly silent and provides little guidance on the optimal age and timing of annuitization similar to the prescriptive implications of portfolio theory. The need for such guidance is especially important given recent proposals to partially privatize Social Security in the U.S. and to allow individuals to manage their pensions, yet force them to annuitize at some pre-determined age, for example age 67.

Therefore, in an attempt to fill this gap, we use preference-free dominance arguments to develop a framework for choosing the optimal age (time) at which to purchase a life annuity, within the structure of *realistic* annuity markets that impose asset-based mortality risk charges. Using annuity mortality tables and the characteristics of real-world annuity payouts – which impose explicit charges for assuming aggregate mortality risk and offer limited selection of investment products available within payout annuities – we show that for most existing products, annuitization prior to age 65-70 is dominated by self-annuitization even in the absence of any bequest motives. For retirees who are willing to accept a minimal level of risk in exchange for retaining the benefits of liquidity, the optimal age can be even higher.

At the very least, economists who are trying to solve the so-called annuity puzzle should be made aware of the precise ages at which this phenomena is rationally puzzling.

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Variable Payout Annuities in the U.S.		
Company Name	# of Investment Funds	Mortality Risk Fee
Hartford Life	30	1.25%
TIAA-CREF	8	0.40%
USAllianz	66	1.25%
American Skandia	93	1.10%
T. Rowe Price	10	0.55%
Fidelity Investments	32	0.75%
Vanguard	15 - 22	0.52%
Nationwide	66	1.25%
AXA Equitable	11	0.40%

Table #1: List of Insurance Companies offering VPAs

Note: Vanguard's 22 choices are for tax-sheltered funds only.

Mortality Rates used for Payout Annuity Pricing		
Current Age (x)	Female q_x	Male q_x
50	0.00154	0.00299
60	0.00386	0.00643
65	0.00625	0.00994
66	0.00688	0.01102
67	0.00756	0.01225
68	0.00829	0.01366
69	0.00910	0.01523
70	0.01003	0.01698
75	0.01756	0.02830

Table #2: Source IAM 2000 Table (www.soa.org)

Variable-Payout Annuity: Initial Income per $100,000

Payments Can Increase or Decrease in Subsequent Years

	AIR = 3%	AIR = 5%	AIR = 7%
Age 65	$6,552.65	$8,020.53	$9600,61
Age 70	$7,639.42	$9,104.15	$10,665.98
D.P.S. (l)	18.1 b.p.	17.8 b.p.	17.4 b.p.

Table #3: The Dominating Portfolio Spread (DPS)

Assuming a Mortality Risk Fee (m) of 80 basis points