Portfolio choice and mortality-contingent claims: The general HARA case

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Abstract

We solve a portfolio choice problem that includes mortality-contingent claims and labor income under general HARA preferences. Our contribution beyond existing literature is to (i) focus on the covariance between shocks to human capital and financial capital, to (ii) model the utility of a family with basic needs, (iii) include life insurance and pension annuity claims in one unified life-cycle model. Our solution employs a "similarity reduction" mapping which reduces the two dimensional HJB equation into one dimension. This allows for the implementation of a quick numerical scheme. And, when shocks to human capital and financial capital are perfectly correlated, a closed-form expression is obtained as a special case.

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1. Introduction

This paper focuses on the decision of how much life insurance and pension annuities a *family unit* should purchase to protect against the loss of its *breadwinner's human capital* as well as how the family should allocate its financial resources between risk-free and risky assets before and after a formal retirement date. In contrast to previous and existing research our framework provides guidance on the optimal demand for pension life annuities as the family anticipates a retirement date of the breadwinner. Pension annuities are the exact opposite of life insurance and are yet another type of mortality-contingent claim.

Our paper is normative and assumes that families have rational preferences that can be specified via hyperbolic absolute risk aversion (HARA) utility. By this HARA specification

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we effectively impose the axiomatic condition that instantaneous consumption rate must be above a lower bound while the breadwinner is alive and after his or her death.

This paper is an extension of Huang, Milevsky and Wang (2006) and is closely related to Pliska and Ye (2007), Bodie, Detemple, Ortuba and Walter (2004) as well as Chen, Ibbotson, Milevsky and Zhu (2005), all of which develop a limited model of insurance demand and/or retirement income planning under more restrictive conditions.

The research falls within the portfolio choice literature, which has been growing in popularity and importance. For example, Sundaresan and Zapatero (1997) examine the role of defined benefit pensions, while Dybvig and Liu (2004), Bodie, Detemple, Ortuba and Walter (2004) model the impact of flexible retirement dates. Jagannathan and Kocherlakota (1996) and Viceira (2001) examine the impact of aging on the demand for risky assets. Koo (1998) as well as Davis and Willen (2001) model the role of labor income. Heaton and Lucas (2000) remind us and provide a rigorous model for the impact and the role of entrepreneurial risk. Others go back to basics and extend portfolio choice models to include more sophisticated (and realistic) processes for investment returns, such as Chacko and Viceira (2005) or time-varying and mean-reverting risk premiums, such as Detemple, Garcia, Rindisbacher (2003,2005).

The unifying theme of the recent portfolio choice literature is that usually a singular feature is carefully modeled in order to provide relevant financial economic insights related to that one feature. On a related note, recent empirical work by Berkowitz and Qiu (2006) has documented that individual health status has an impact on portfolio holdings over the life-cycle, but perhaps not to the extent previously believed in the literature. We develop a normative model that optimizes portfolio holdings – with insurance and pensions – to account for mortality risk.

Our methodology uses a more general HARA utility of consumption as opposed to the (easier, and) more restrictive CRRA model. Similar to the earlier work by Huang, Milevsky and Wang (2006), we deviate somewhat from previous models by focusing on the *family unit* that derives utility from consumption. Our main concern is the risk to this family of losing the wage/income source. The emphasis on the family as a unit, which survives independently of the life status of the breadwinner, alleviates the need for specifying a separate utility of bequest. [1]

Life insurance has in fact been previously analyzed in a financial context. Yaari (1965), Hakanson (1969), Fischer (1973), Campbell (1980), Lewis (1989), Hurd (1989) and Babbel and Ohtsuka (1989) develop a number of insights into the demand for life insurance. These papers do not give us much insight into the role of labor income dynamics and human capital over the life-cycle. The same is true of the work by Richards (1975), which extended Merton (1969) to life insurance. Along the same lines, Buser and Smith (1983) adopt a

[1] This is different from utility of consumption versus bequest approach taken in Pliska and Ye (2007). Likewise, our work differs from the recent paper by Chen, Ibbotson, Milevsky and Zhu (2005) where (i) the analysis is done in discrete time, (ii) consumption is assumed exogenous, and (iii) the analysis is done completely via simulations.

portfolio approach to life insurance, but from a strictly one-period framework. We bring all the different strands together in one unified framework by allowing both long positions in (i.e. purchases of) life insurance as well as short positions in (i.e. the sale of) life insurance, which is effectively a long position in an instantaneous pension annuity. We find that even a few years prior to the retirement date, the breadwinner stops holding life insurance and starts receiving income from a pension annuity.

>From a technical point of view, this two state variable class of problems – wage income and asset prices – normally leads to a highly nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation. Despite this nonlinearity, analytical solutions can be obtained in special cases when wage income and asset prices are perfectly (positively or negatively) correlated. When this correlation is between -1 and 1, we use the method of *similarity reduction* to reduce the dimension of the HJB equation before applying a simple finite difference method. As a result, the computational cost is greatly reduced, compared to that of a full numerical approach applied to the original two-dimensional HJB. See for example Wang (2006) for a description of the more cumbersome (traditional) approach to solving such two dimensional problems.

The rest of the paper is organized as follows. The setup of the problem is described in Section 2, followed by the setup of the HJB equation and solution methodologies in Section 3. In Section 3.2, we focus on the closed-form solution and the numerical method is discussed in Section 3.3. In Section 4 some results and the financial implications of these results are presented. We summarize the paper in Section 5. The derivations of the HJB equation and other non-essential derivations are placed in the special appendices. [2]

2. Stochastic model

The variable t denotes the current time and we will work with three dates of interest. The first date is the time horizon T_H of the family, which is assumed exogenous and deterministic on the order of 50 to 100 years. The second date is the time of retirement $T_R < T_H$, which is when the wage/income process (job) terminates and the breadwinner enters his or her retirement years. The third date of interest is the death of the breadwinner, and end of the wage/income process, which takes place at a random stopping time denoted by τ. The wage/income process jumps to zero at the minimum of T_R and τ. We do not allow flexible retirement as in Sundaresan and Zapatero (1997).

We assume the family purchases short-term insurance on the life of the breadwinner, which is renegotiated and guaranteed renewable on an ongoing basis at a predetermined schedule which is driven by an *instantaneous force of mortality (IFM) curve* denoted by λ_{y+t}, where y is the age of the primary breadwinner at inception of the model. We then let I_t denote the insurance premium (in dollars) payable per unit time, a variable which is under the direct

[2] This is available from the authors upon request as part of the original, longer, working paper. It can also be downloaded from www.ifid.ca

control of the family. One can think of I_t as a budget for insurance, which will then induce a certain face value or death benefit I_t/λ_{y+t}. For example, if the IFM curve at age $y = 35$ is $\lambda_{35} = 0.001$, and the family spends $I_0 = 50$ dollars on life insurance, this entitles the family to a death benefit of $50/0.001 = 50$ thousand dollars.

The family is also allowed to purchase a pension annuity on the life of the breadwinner. To make the problem simple, we assume that the annuity can be treated as a negative insurance policy; instead of paying the premium, the family receives an annuity payment, also indicated by I_t. Upon the death of the breadwinner at time t, the family has to pay back the face value of the annuity policy in a lump sum $I_t/\bar{\lambda}_{y+t}$. We note that in reality, the face value is usually paid up-front, instead of at the time of death. Therefore, $\bar{\lambda}_{y+t}$ may be different from λ_{y+t}. However, in this paper we will assume that the two rates are equal for simplicity.

We let M_t denote the market value of the family's assets, which includes the value of all (risky) stocks and (risk-free) bonds on a mark-to-market basis. We assume that M_0 denotes the initial marketable wealth at time $t = 0$. Heuristically, the breadwinner works and converts labor and time into wages and income. A portion of this income is consumed and the remainder is saved in a diversified portfolio consisting of risky stocks and safe bonds.

The variable α_t denotes the fractional allocation of the family's marketable wealth M_t to the risky asset at time t. Thus, if $\alpha_t = 0$ the family allocates all its marketable wealth – but no more – to risk-free bonds. Conversely if $\alpha_t = 1$ the family allocates all its marketable wealth – but no more – to risky stocks. The model also allows for $\alpha_t > 1$ which would imply leverage. Thus, for example, $\alpha_t = 2$ implies that 200% of wealth is invested in the risky stock. This is financed by borrowing 100% of wealth at some (constant) rate of interest denoted by r. Recall that α_t is under direct control of the family and is one of the *three* choice variables in our model. We do not address liquidity constraints and the difficulty faced by younger families when borrowing. Note that many of our numerical results involve large amounts of leverage at young ages. This is not unique to the HARA model.

We let c_t denote the instantaneousness consumption rate of the family (in real terms) per unit time. In general, our model is specified in real (after inflation) terms and all parameters and choice variables will reflect this. Note that the consumption rate is our third and final choice (a.k.a. control) variable and c_t is chosen to maximize the family's *utility* of consumption.

Even though a more realistic state-dependent instantaneous utility function can be used, under our PDE-based methodology, we adopted a simpler HARA utility function in this study. The precise functional form is

$$u(c) = \begin{cases} u_l(c) \equiv \frac{1}{1-\gamma}(c - c_l)^{1-\gamma} & \text{t} < \tau \\ u_d(c) \equiv \frac{1}{1-\gamma}(c - c_d)^{1-\gamma} & \text{t} \geq \tau \end{cases} \tag{1}$$

for some positive constant $\gamma > 0$, which is the coefficient of relative risk aversion. c_l and c_d are the consumption floors for the family, before and after the death of the breadwinner. Here τ is the random time of death. The specification of a consumption floor, both prior to

– and after – death is one of the main aspects that differentiates this paper from Huang, Milevsky and Wang (2006), where only CRRA utility is allowed.

Let X_t denote the market value of the risky asset (stock index, market portfolio) at time t. This stochastic process will be modeled as a geometric Brownian motion so that:

$$dX_t = \mu_m X_t dt + \sigma_m X_t dB_t^m, \tag{2}$$

where μ_m denotes the drift and σ_m denotes the diffusion coefficient of the process. This, of course, implies that $\ln(X_t/X_0)$ is normally distributed with a mean value of $(\mu_m - 0.5\sigma_m^2)t$ and a standard deviation of $\sigma_m\sqrt{t}$. Stated differently, the geometric mean return (a.k.a. growth rate of the risky asset is $\mu_m - 0.5\sigma_m^2$ per annum. Typical values of the parameters μ_m fall in the range of $(5\%, 15\%)$ and typical values of σ_m fall in the range of $(5\%, 50\%)$. The risk-free rate r, which is also the rate at which the family can borrow money, is on the order of magnitude of $(1\%, 3\%)$, which must be lower than μ_m for economic equilibrium purposes.

We now let W_t denote the real (after-inflation) wage/income rate of the family's bread-winner per unit time. This stochastic process is expected to increase in real terms over time and might be correlated with the investment performance of the risky asset. We assume the wage process satisfies the following geometric Brownian motion (GBM), but with specification:

$$dW_t = \begin{cases} \mu_w W_t dt + \sigma_w W_t dB_t^W & t < \min[\tau, T_R] \\ 0 & t \geq \min[\tau, T_R] \end{cases}, \tag{3}$$

where μ_w denotes the drift and σ_w denotes the diffusion coefficient of the process, and B_t^W denotes the Brownian motion driving the wages process W_t. Similar to the risky asset, we assume that the real wage at any future time $t+s$ is lognormally distributed with parameters (μ_w, σ_w). We could just as easily specify a mean-reverting process. Note that the Brownian motion B_t^m driving the risky asset is instantaneously correlated with the B_t^W driving the wage process via the relationship $d\left\langle B_t^m, B_t^W \right\rangle = \rho\sigma_m\sigma_w dt$. Later we will talk about this correlation variable ρ, which is the primary focus of our numerical case study and results.

As mentioned earlier, the instantaneous force of mortality (or hazard rate) is denoted by λ_{y+t}, where y is the initial age. The quantity $\lambda_{y+t}dt$ can be thought of as the rate of death within a small time interval dt at time t. The conditional probability of survival, from age y to age $y+t$, under the law of mortality defined by λ_{y+t} can be computed via:

$${}_tp_y = e^{-\int_0^t \lambda_{y+s} ds}, \tag{4}$$

where the notation on the left-hand side of the equation is standard in the actuarial literature. For example, if the future lifetime random variable is exponentially distributed, we have that $\Pr[\tau \geq s] = e^{-\lambda s}$, and the hazard rate is constant at all ages, at a value of $\lambda_{y+t} = \lambda$. In the general case, the function λ_{y+t} is expected to increase with time (age). It is important to stress that in our model the family unit knows exactly how much they will have to pay for insurance – regardless of how much and when they want to purchase it – from the current time t, to the time of retirement T_R. Thus, in addition to precluding whole-life and other more complicated forms of insurance, we do not allow for stochastic mortality rates or anti-selection effects which might complicate the insurance purchase problem. Thus, once again, if the family purchases (invests) I_t dollars in life insurance at time t, they will be entitled to a death benefit of I_t/λ_{y+t} if the breadwinner dies at time t. For now, we ignore loading and commissions, which can easily be handled by working with a loaded hazard rate $\widehat{\lambda}_{y+t}$ instead of a biological mortality rate λ_{y+t} as shown in a special appendix.[3]

Based on the construction of the wage/income process W_t and the evolution of the risky asset price X_t, the family budget constraint for the marketable wealth process M_t will satisfy the following stochastic differential equation:

$$\begin{aligned} dM_t &= W_t dt - c_t dt - I_t dt + \alpha_t M_t(\mu_m dt + \sigma_m dB_t^m) + (1-\alpha_t) r M_t dt \\ &= [(\mu_m - r) M_t \alpha_t + r M_t + W_t - c_t - I_t]\, dt + \sigma_m \alpha_t M_t dB_t^m \end{aligned} \tag{5}$$

for $t < \min[\tau, T_R]$, and

$$\begin{aligned} dM_t &= -c_t dt + \alpha_t M_t(\mu_m dt + \sigma_m dB_t^m) + (1-\alpha_t) M_t r dt \\ &= [(\mu_m - r) M_t \alpha_t + r M_t - c_t]\, dt + \sigma_m \alpha_t M_t dB_t^m \end{aligned} \tag{6}$$

for $t > \min[\tau, T_R]$. To simplify the problem, we have ignored the pension after the retirement either in the form of a jump in the wealth at retirement or in the form of pension income after retirement, which will be addressed in future work.

The intuition for the various pieces in equations (5) and (6) is as follows. First, we add wage income – when there is some – via $W_t dt$. Then, we subtract discretionary consumption $c_t dt$ and insurance premiums $I_t dt$. Finally, we add instantaneous investment returns from the allocation to the risky asset, $\alpha_t M_t dX_t/X_t$ as well allocations to the risk-free asset $(1 - \alpha_t) M_t r dt$. Finally, we have substituted dX_t/X_t from equation (2) to eliminate any reference to X_t from here on.

At the instant of death $t = \tau$, we must carefully add the death benefit/annuity face value by distinguishing between the value of marketable wealth one instant prior to the "arrival" of death and one instant after this arrival. It is worth noting that the wealth process before death implicitly depends on a Poisson process, and the first jump of this process occurs at death. The family of the deceased (or the estate) will receive a lump sum payment from the insurance company at the time of death, so the wealth process of the family will obviously jump at that time as well. More precisely:

$$M_{\tau+} = M_{\tau-} + \frac{I_\tau}{\lambda_{y+\tau}} \tag{7}$$

where $I_\tau/\lambda_{y+\tau}$ is the death benefit or the face value of the annuity policy at time τ, $\lambda_{y+\tau}$ is the instantaneous force of mortality, or hazard rate. As we stated earlier, we assume that

[3] This is available from the authors upon request as part of the original, longer, working paper. It can also be downloaded from www.ifid.ca

the hazard rate for the annuity policy is the same as that for the life insurance for simplicity in this paper.

Finally, the family-unit or household objective function is defined by:

$$J = \max_{\{\alpha_s, I_s, c_s\}} \left\{ E_t \left[\int_t^{T_H} e^{-\delta s} u(c_s) ds | \mathcal{F}_t \right] \right\}, \tag{8}$$

where δ is the subjective discount rate and $u(c_s)$ denotes the utility function (in the form of $u_l(c_s)$ or $u_d(c_s)$ depending on whether the breadwinner is alive or not). In words, the family is searching for an asset allocation strategy α_s, an insurance buying strategy I_s, and consumption strategy c_s that maximizes the discounted value of utility of consumption between time t (now) and the terminal horizon of the family unit.

We stress once again that the main difference between life and death is that we have specified the c_l and c_d levels that must be achieved in all states. This HARA model, drives a number of the interesting results regarding short positions in life insurance (i.e. long positions in pension annuities) as the family approaches retirement.

3. Control strategy

We now discuss the control strategy – which determines the joint holding of mortality contingent claims as well as the exposure to risky portfolio assets – using the dynamic programming principle. We will examine the associated Hamilton-Jacobi-Bellman (HJB) equation and related closed-form solutions first. We will also discuss a numerical method based on the concept of a similarity reduction.

3.1. Hamilton-Jacobi-Bellman (HJB) equation

First we use $\Phi(M, t)$ to denote the solution after the death of the breadwinner and it satisfies the following HJB (without wage and insurance)

$$\begin{aligned} 0 = \Phi_t &+ \max_{c_t} \left(e^{-\delta t} u_d(c_t) - c_t \Phi_M \right) \\ &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t \Phi_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 \Phi_{MM} \right] + r M_t \Phi_M. \end{aligned} \tag{9}$$

The HJB equation for the portfolio choice problem can be derived as follows:

$$\begin{aligned}\lambda_{y+t}J = {} & J_t + \max_{c_t}\left(e^{-\delta t}u_l(c_t) - c_tJ_M\right) \\ & + \max_{I_t}\left(\Phi\left(M_{t+},t\right)\lambda_{y+t} - I_tJ_M\right) \\ & + \max_{\alpha_t}\left[\alpha_t(\mu_m - r)M_tJ_M + \frac{1}{2}\alpha_t^2\sigma_m^2M_t^2J_{MM} + \alpha_t\rho\sigma_w\sigma_mW_tM_tJ_{WM}\right] \\ & + rM_tJ_M + W_tJ_M + \mu_wW_tJ_W + \frac{1}{2}\sigma_w^2W_t^2J_{WW} \end{aligned} \tag{10}$$

for $t < T_R$ and

$$\begin{aligned}\lambda_{y+t}J = {} & J_t + \max_{c_t}\left(e^{-\delta t}u_l(c_t) - c_tJ_M\right) \\ & + \max_{I_t}\left(\Phi\left(M_{t+},t\right)\lambda_{y+t} - I_tJ_M\right) \\ & + \max_{\alpha_t}\left[\alpha_t(\mu_m - r)M_tJ_M + \frac{1}{2}\alpha_t^2\sigma_m^2M_t^2J_{MM}\right] + rM_tJ_M. \end{aligned} \tag{11}$$

for $t \geq T_R$. Since there is zero-bequest, the terminal conditions are

$$J(M,W,T_H) = \Phi(M,T_H) = 0. \tag{12}$$

These equations are standard in the portfolio choice literature. The non-conventional term (at least for those not familiar with insurance) is the hazard rate λ_{y+t}, which is on the left-hand side of equation (11) and allows for people to actually die.

3.2. Closed form solutions

3.2.1. Post-death period

After the death of the breadwinner, the solution is given by Φ, which can be obtained (following Merton, 1969) by writing it in the form

$$\Phi = \frac{h(M - g_d)^{1-\gamma}}{1-\gamma}$$

where h and g_d satisfy the following equations

$$\frac{1}{1-\gamma}\frac{h'}{h} + \frac{\gamma}{1-\gamma}e^{-\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}} + r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2} = 0, \tag{13}$$

$$g_d' - rg_d + c_d = 0 \tag{14}$$

with the terminal conditions (at T_H)

$$h(T_H) = 0,\ g_d(T_H) = 0.$$

The solutions for h and g_d can be obtained easily as

$$h(t) = e^{-\delta t}\left(\frac{e^{\eta_h(T_H - t)} - 1}{\eta_h}\right)^{\gamma}, \tag{15}$$

$$g_d(t) = \frac{c_d}{r}\left(1 - e^{-r(T_H - t)}\right), \tag{16}$$

with

$$\eta_h = -\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}\left(r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2}\right).$$

The optimal controls are given as

$$c^* = c_d + e^{-\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}}(M - g_d), \tag{17}$$

$$\alpha^* = \frac{\mu_m - r}{\gamma\sigma_m^2}\left(1 - \frac{g_d}{M}\right). \tag{18}$$

In words, we work our way backwards from the point at which the breadwinner has died and hence the family is (only then) faced with a conventional portfolio choice problem. In addition to their intermediate role in the solution, here is how to think of the g_d variable/quantity from a financial economic perspective. Basically, the optimal policy – both in terms of the demand for insurance and the allocation to the risky asset – is driven by the need to satisfy the minimal consumption level c_d, which is required by the family once the breadwinner is deceased. The g_d can be viewed or treated as a type of present of discounted value of this required level of consumption over the horizon of the family, but only after the death of the breadwinner. Indeed, if the time horizon $T_H = \infty$, and the family is concerned with perpetual consumption, then $g_d = c_d/r$, which is the discounted value of perpetual consumption, or the market value of a console bond at the rate of r. Once the family has this level of wealth, it can secure or lock-in the floor by investing g_d in the risk-free asset.

3.2.2. Pre-death post-retirement period

When the breadwinner is retired but still alive, the solution is given by J, which can be found in a similar way as the post-death period. We apply the first order conditions, which yields the following equation

$$\begin{aligned}\lambda_{y+t}J = J_t &+ \frac{\gamma}{1-\gamma}(c^* - c_l)J_M + \frac{\gamma\lambda_{y+t}}{1-\gamma}(M + \lambda_{y+t}^{-1}I^* - g_d)J_M \\ &+ \lambda_{y+t}(M - g_d)J_M - c_l J_M + rMJ_M + \frac{\alpha^*}{2}(\mu_m - r)MJ_M\end{aligned} \tag{19}$$

where the optimal control functions are given by the first order conditions:

$$c^* = e^{-\frac{\delta}{\gamma}t}(J_M)^{-\frac{1}{\gamma}} + c_l, \tag{20}$$

$$\lambda_{y+t}^{-1} I^* = h^{\frac{1}{\gamma}}(J_M)^{-\frac{1}{\gamma}} - M + g_d, \tag{21}$$

$$\alpha^* = -\frac{(\mu_m - r)J_M}{\sigma_m^2 M J_{MM}}. \tag{22}$$

We seek the solution in the form

$$J = \frac{h(M - g_l)^{1-\gamma}}{1-\gamma}$$

where g_l is a function of time t satisfying

$$g_l' - r g_l - \lambda_{y+t}(g_l - g_d) + c_l = 0. \tag{23}$$

The terminal condition for g_l is

$$g_l(T_H) = 0. \tag{24}$$

Instead of solving g_l directly, we will work on the difference $g = g_l - g_d$. From the equations for g_d and g_l, we can obtain the equation for $g = g_l - g_d$ as

$$g' - (r + \lambda_{y+t})g = -\Delta_c \tag{25}$$

where $\Delta_c = c_l - c_d$. This equation can be integrated out as

$$g = -\Delta_c e^{\int_{T_H}^t (r+\lambda_{y+s})ds} \int_{T_H}^t e^{-\int_{T_H}^z (r+\lambda_{y+s})ds} dz. \tag{26}$$

The optimal controls are now

$$c^* = c_l + e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}}(M - g_d - g), \tag{27}$$

$$\lambda_{y+t}^{-1} I^* = -g, \tag{28}$$

$$\alpha^* = \frac{\mu_m - r}{\gamma\sigma_m^2}\left(1 - \frac{g_d + g}{M}\right). \tag{29}$$

3.2.3. Pre-death pre-retirement period

In this case, the situation is more complicated compared to the other two end-points. The closed form solution can be obtained only for the special case of $\rho = \pm 1$, i.e., shocks to wages and risky returns are perfectly correlated or anti-correlated. After applying the first order conditions to the HJB, we obtain

$$\lambda_{y+t}J = J_t + \frac{\gamma}{1-\gamma}(c^* - c_l)J_M + \frac{\gamma\lambda_{y+t}}{1-\gamma}(M + \lambda_{y+t}^{-1}I^* - g_d)J_M$$
$$+ \lambda_{y+t}(M - g_d)J_M - c_l J_M + rMJ_M + WJ_M + \mu_w WJ_W + \frac{1}{2}\sigma_w^2 W^2 J_{WW}$$
$$+ \frac{\alpha^*}{2}(\mu_m - r)MJ_M + \frac{\alpha^*}{2}\rho\sigma_m\sigma_w MWJ_{MW} \tag{30}$$

where the optimal control functions are given by the first order conditions:

$$c^* = e^{-\frac{\delta}{\gamma}t}(J_M)^{-\frac{1}{\gamma}} + c_l, \tag{31}$$
$$\lambda_{y+t}^{-1}I^* = h^{\frac{1}{\gamma}}(J_M)^{-\frac{1}{\gamma}} - M + g_d, \tag{32}$$
$$\alpha^* = -\frac{(\mu_m - r)J_M + \rho\sigma_m\sigma_w WJ_{MW}}{\sigma_m^2 MJ_{MM}}. \tag{33}$$

We look for the solution in the form of:

$$J = \frac{h(M + kW - g_l)^{1-\gamma}}{1-\gamma}$$

where h and g_l are as defined earlier and the k function depends on time t only. From an economic point of view, k can be viewed as the time-adjustment that is made to the wage process, both in terms of adjusting for financial risk and for mortality risk. In other words, given an annual wage income of W at time t, $k(t)W$ can be viewed as the "total value" of the wage income (from t until death or retirement). After algebraic manipulations (details available from author), we obtain that the function k satisfies:

$$\frac{k'}{k} + \frac{1}{k} + \mu_w - r - \lambda_{y+t} - \beta(\mu_m - r) = 0, \tag{34}$$

Since the wage stops at retirement, we have the terminal condition for k as

$$k(T_R) = 0. \tag{35}$$

The first order equation for k can be solved as

$$k = e^{-\int_{T_R}^t \eta_k(s)ds}\int_{T_R}^t e^{-\int_{T_R}^s \eta_k(t)dt}ds, \tag{36}$$

for $t < T_R$ and $k = 0$ for $T_R \leq t \leq T_H$. Here $\eta_k = \mu_w - r - \lambda_{y+t} - \beta(\mu_m - r)$ and $\beta = \rho\sigma_w/\sigma_w$.

The optimal controls are now

$$c^* = c_l + e^{-\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}}(M + kW - g_d - g), \tag{37}$$
$$\lambda_{y+t}^{-1}I^* = kW - g, \tag{38}$$
$$\alpha^* = \frac{\mu_m - r}{\gamma\sigma_m^2}\left(1 + \frac{kW}{M} - \frac{g_d + g}{M}\right) - \beta\frac{kW}{M}. \tag{39}$$

3.2.4. Discussion: life insurance or pension annuity?

Based on our closed-form solution, the optimal demand (either positive or negative) for mortality-contingent claims is given by:

$$\lambda_{y+t}^{-1} I^* = kW - g \tag{40}$$

when $\rho = \pm 1$.

Note that this expression has two components. The first term is the same as in the case of CRRA utility – see for example Pliska and Ye (2007) or Huang, Milevsky and Wang (2006) – which is the protection against wage income. The second term is purely driven by the extra HARA constant in the utility function and determines how to maintain the minimal level of consumption the family desires, both before and after death.

There exists a critical wage level $W_c = k^{-1}g$. When $W \geq W_c$, life insurance is desirable and demanded, but when the wage level is lower than W_c, the pension annuity is preferred. This does not occur under CRRA utility as there is no difference in consumption levels. In other words, the specification of a minimal consumption floor forces people to start selling insurance prior to the retirement date. Figure #1 (which will be presented and discussed later) illustrates how this might evolve over typical parameter values.

The wealth level itself does not appear explicitly in the insurance/annuity demand or cost. Instead, it is determined by the instantaneous wage as well as the difference between g_d and g_l, which are the amounts of wealth set aside to guarantee the consumption needs specified by the floors c_d and c_l.

We now examine several special cases to show how these consumption floors affect the value of g.

3.2.5. Uniform consumption floors: $c_l = c_d$

When $c_d = c_l$, we have $\Delta_c = 0$. Thus $g \equiv 0$, or $g_l \equiv g_d$. In this case the optimal insurance payment is simply:

$$\lambda_{y+t}^{-1} I^* = kW, \tag{41}$$

which is the same as the CRRA. Thus, the level of consumption floor has no impact on the optimal amount of insurance. No annuity is necessary. In this case, the wealth level also has no effect.

3.2.6. Different consumption floors: $c_l > c_d$

When the consumption floors are different, $g > 0$, therefore the level of optimal insurance is lower (annuity is higher). The actual amount is also affected by the mortality rate and below are two typical examples.

(i) *Constant* λ_{y+t}. When $\lambda_{y+t} = \lambda$. In this case we have:

$$g = \frac{c_l - c_d}{r + \lambda} \left(1 - e^{-(r+\lambda)(T_H - t)}.\right) \tag{42}$$

(ii) *Gompertz mortality.* A more realistic mortality rate is the Gompertz rate:

$$\lambda_{y+t} = \lambda_0 + \frac{1}{b} e^{\frac{y+t-m}{b}}. \tag{43}$$

In this case the solution for g can be worked out as

$$\begin{aligned} g = &(c_d - c_l) b e^{-(r+\lambda_0)(y+t-m) + e^{\frac{y+t-m}{b}}} \\ &\times \left[\Gamma\left(-(r+\lambda_0)b, e^{\frac{y+T_H-m}{b}}\right) - \Gamma\left(-(r+\lambda_0)b, e^{\frac{y+t-m}{b}}\right)\right]. \end{aligned}$$

Here the incomplete gamma function is defined as

$$\Gamma(a, b) = \int_b^\infty t^{1-a} e^{-t} dt.$$

3.3. Numerical solution

For more general cases, we must use numerical means to find approximate solutions. Obviously one can apply a numerical method to the two dimensional HJB equation directly, and this is the approach taken and discussed by Wang (2006). However, as was pointed out by Huang, Milevsky and Wang (2006) in the case of CRRA utility, if one can locate a similarity variable and reduce the dimension of the equation, numerical solutions can be obtained more efficiently. A helpful technique is to take the ratio between the wage W and the wealth M as the main variable and then apply the similarity reduction technique. Here we use a more general similarity variable:

$$x = \frac{W}{\hat{M}} \tag{44}$$

where $\hat{M} = M + kW - g_l(t)$ is the adjusted wealth including the human capital portion (kW). We then seek solution in the form of:

$$J = \frac{\hat{M}^{1-\gamma} F(x,t)}{1-\gamma}. \tag{45}$$

Note that $\hat{M}$ is always positive. Thus, the approach used here is valid for all financially relevant scenarios, including the case of $M < g_l$ or even $M < 0$. After algebraic manipulations, we obtain a simplified one-dimensional equation for F (details available from the authors upon request.)

$$0 = F_t + c_0 F + c_1 x F_x + c_2 x^2 F_{xx} \tag{46}$$

where

$$
\begin{aligned}
c_0 = & -\lambda_{y+t} + (1-\gamma)(\mu_m - r)\beta kx + \gamma(\hat{c} + \lambda_{y+t}\hat{I}) \\
& + (1-\gamma)(\lambda_{y+t} + r) + (1-\gamma)\hat{\alpha}(\mu_m - r) && (47) \\
& - \frac{\gamma(1-\gamma)}{2}\left[(\sigma_w kx + \rho\hat{\alpha}\sigma_m)^2 + (1-\rho^2)(\hat{\alpha}\sigma_m)^2\right], && (48) \\
c_1 = & -(\mu_m - r)\beta kx - \frac{\gamma}{1-\gamma}(\hat{c} + \lambda_{y+t}\hat{I}) - (\lambda_{y+t} + r) - \hat{\alpha}(\mu_m - r) \\
& + \frac{\gamma}{2}\left\{(\sigma_w kx + \rho\hat{\alpha}\sigma_m)^2 + [\sigma_w(kx-1) + \rho\hat{\alpha}\sigma_m]^2 + 2(1-\rho^2)(\hat{\alpha}\sigma_m)^2\right\} && (49) \\
& + \mu_w, && (50) \\
c_2 = & \frac{\gamma}{2}\left\{[\sigma_w(kx-1) + \rho\hat{\alpha}\sigma_m]^2 + (1-\rho^2)(\hat{\alpha}\sigma_m)^2\right\}. && (51)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{c} &:= \frac{c^* - c_l}{\hat{M}} = e^{-\frac{\delta}{\gamma}t} G^{-\frac{1}{\gamma}}, && (52) \\
\hat{I} &:= \frac{\lambda_{y+t}^{-1} I^* + M - g_d}{\hat{M}} = h^{\frac{1}{\gamma}} G^{-\frac{1}{\gamma}}, && (53) \\
\hat{\alpha} &:= \frac{\alpha^* M}{\hat{M}} = \frac{\frac{\mu_m - r}{\sigma_m^2} G + \beta x G_x}{\gamma G + x G_x} - \beta kx, && (54)
\end{aligned}
$$

and

$$
G = F - \frac{xF_x}{1-\gamma}, \quad H = kG + \frac{F_x}{1-\gamma}.
$$

At retirement T_R, we have

$$
F(x, T_R) = h(T_R). \tag{55}
$$

The original solution can be recovered by using (45).

Remark. Note that in the case of $x = 0$, we have $F_x = 0$, $G = F$ and $H = kF$. Equation (46) reduces to (13), which yields $F \equiv h$, and we recover Merton's solution. Another special case is when $\rho = \pm 1$, in which case we also have $F \equiv h$, corresponding to the closed form solution obtained in Section 3.2.3..

Equation (46) must be solved numerically. In this paper, we use a one-step backward Euler method for the time discretization and central and upwind difference formulas to approximate F_{xx} and F_x, respectively. Our method is semi-implicit since controls (52)-(54) are computed using the solution at the previous time step. The semi-infinite domain $0 < x < \infty$ is truncated into a finite one $0 \leq x \leq x_{max}$. On the computational boundaries $x = 0$ and $x = x_{max}$, we apply the asymptotic relation $F_x = 0$. The discrete system for F is then solved using a so-called tri-diagonal solver.

4. Numerical results and discussion

We now present some case studies or practical recommendations for a realistic set of economic parameters. And so, the parameter values we have chosen for our extensive results are as follows: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$, $M_0 = 1,200,400$ thousand dollars, $W_0 = 50$ thousand dollars, $T_R = 10, 20, 30$ years, $T_H = 40, 50, 60$ years, $\delta = 2\%$, $y = 35, 45, 55$ years.

The financial economic justification for these numbers are as follows. We assume that the family can invest their financial wealth in a broadly diversified index fund that is expected to earn an inflation-adjusted arithmetic mean of $\mu_m = 7\%$ per annum, with a volatility of $\sigma_m = 20\%$. These number are consistent with other asset allocation research in the literature, see for example Campbell and Viciera (2002) or Chen, Ibbotson, Milevsky and Zhu (2005). The risk free rate of $r = 2\%$ after inflation is also consistent with current conditions in the (inflation-adjusted) bond market. We further assume that wages will grow in real (after inflation) terms by approximately $\mu_w = 1\%$ per annum, with a standard deviation of $\sigma_w = 5\%$. This is consistent with wage inflation being higher than price inflation, but also allows for shocks to wages, such as unemployment or business cycle effects. Finally, we assume the family's discount rate – or subjective rate of time preference – is $\delta = r = 2\%$, which is consistent with various macroeconomic models as well as the work by Pliska and Ye (2006), within the context of insurance.

In terms of other embedded parameters, we assume that insurance prices (and mortality rates) are driven by a Gompertz law of mortality, which at age 45 starts at: $\lambda_{45+t} = \exp\{(45+t-86.3)/9.5\}/9.5$. These numbers are consistent with survival rates implicit in pension-based mortality tables and the analytic law of mortality has been used in a variety of actuarial and insurance papers.

Since $M + kW - g_l$ must be non-negative, it implies that there exists a maximum pre-death consumption floor c_l. In addition, it is financially sensible to assume that the post-death consumption floor c_d should be less than c_l. In the first three numerical examples, we have used three different consumption levels: $c_d = 0$ (CRRA), \$10 and \$20 thousands. [4]

Table #1 Placed Here

More specifically, Table #1 displays the optimal allocation to risky (market) stocks and the optimal face amount (death benefit) for life insurance, assuming the primary bread-winner has $T_R = 20$ years to retirement and the family horizon is $T_H = 50$ years. This individual is $y = 45$ years-old, earns $W_0 = 50$ thousand dollars per year in real (after inflation) terms and

[4] We have chosen a maximum floor of $c_d = \$20$ thousands for all the computations since this value is close to the maximum allowed value for one of the cases considered, based on the estimate using $M_0 + k(0)W_0 > g_l(0)$. This estimate is based on the assumption that the pre-death consumption floor is 20% more that the post-death one.

currently has $M_0 = 200$ thousand dollars saved. These numbers are fairly realistic when we consider the fact that a 45 year-old should already have some amount of savings set aside for retirement purposes.

Notice that to be consistent with how insurance is discussed by industry practitioners, we have chosen to display the actual face value of life insurance I_t^*/λ_{y+t}, instead of the amount spent on life insurance I_t^* or the ratio of wealth insurance I_t^*/M_t.

Table #2 provides similar results, but with the key difference that the individual is now $y = 55$ years-old and therefore $T_R = 10$ years from retirement and $T_H = 40$ years from the terminal horizon of the family unit. We increase the wealth level to $M_0 = \$400$ thousands and assume all other parameters are exactly the same to focus on the impact of "aging" on the optimal face value (i.e. death benefit) of life insurance and the optimal asset allocation.

Table #2 Placed Here

Table #3 takes us back to age 35, which is $T_R = 30$ years prior to retirement and $T_H = 60$ years prior to the end of the family's planning horizon. One other change in Table #3 is that we have reduced the initial (current) wealth $M_0 = 1$ thousand in contrast to the larger values of M_0 of the previous two tables. This particular parameter assumption is motivated by the low levels of wealth one would expect to observe amongst (young) 35 year-old individuals.

Table #3 Placed Here

Figure #1 provides a graphical illustration of the relationship between age, wages and the demand for life insurance or pension annuities. The critical wage curve is the point above which insurance is purchase and bellow which insurance is sold (i.e. pension annuities are purchased.) At a young age the critical wage curve is quite low and as the breadwinner approaches retirement, the curve increases and then asymptotes to infinity by retirement. Recall that when the instantaneous wage is above the critical wage curve insurance is purchased and therefore at a young age one is likely to have positive insurance holdings. But just prior to retirement the realized wage process is likely to fall under the critical wage curve, in which case the breadwinner starts to purchase pension annuity units.

Figure #1 Placed Here

We note that the results reported here are obtained using the numerical procedure outlined in the above section entitled numerical methods. We have run extensive tests to verify our numerical solution using grid refinement as well as comparing numerical solution with the closed form solution for special cases when $\rho = \pm 1$.

Now, instead of examining ages 35, 45 and 55, under "fair" pricing, wo now illustrate the effect of a loaded hazard rate $\hat{\lambda}_{y+t}$. In Table #4, we compare the results at age 45, obtained using the modified HJB equation $\hat{\lambda}_{y+t} = (1 + f)\lambda_{y+t}$ with $f = 0$ and $f = 15\%$.

Table #4 Placed Here

It can be seen that using a loaded hazard rate does have some effect on the solution. In particular[5], the death benefit now changes with the risk aversion γ, even when $\rho = \pm 1$. However, the effect is modest and the values of death benefit do not differ significantly from the case when loading is ignored. What this means is that although theoretically the loading factor will create a dependence between the quantity insured *vis a vis* the risk aversion of the family unit, on a relative basis the impact is small.

For example, imagine a family that has an accumulated net worth of $200 thousands, where there breadwinner is 45 years of age, making $50 thousands per year. The breadwinner plans to work for 20 more years and then retire at the age of 65. The family, on the other hand, needs/desires $20 thousands per year at a minimum if the breadwinner were to die, and $24 thousands while the breadwinner is alive. According to the model results displayed in Table #4, under a risk aversion of $\gamma = 1.5$, the family buys $700 thousands face value life insurance when the price is "fair" (i.e. not loaded) and $671 thousands of face value life insurance when the pricing mortality rate is loaded by 15%. This is a difference of $29 thousands in life insurance, but is less than 5% of the amount demanded. Likewise, the amount invested/allocated to the risky asset changes from 58.8% of the family's net worth, to 56.5% of the family's net worth, when the insurance price is loaded by 15%. In sum, we do not see this as a substantial qualitative difference.

5. Conclusion

Our primary research question was to investigate the interaction between the demand for life insurance, pension annuities, risky assets and the optimal level of consumption as a function of one's occupation, all within a HARA framework. Our main conclusion is that HARA utility – which is likely a more realistic model of family behavior and family desires – makes a difference on the qualitative behavior of the family.

Our model confirms that the optimal amount of life insurance (and pension annuity) a family holds, depends on both the volatility of the wage/income process as well as its correlation to investment returns. We find that higher volatility and higher correlations lead to lower, optimal, life insurance levels.

Finally, our model validates the wisdom that the optimal amount of life insurance a family should hold is not (very) sensitive to the risk aversion of the family, even when a loaded hazard rate for the death benefit is used. Indeed, although our numerical results illustrate that risk aversion (as measured by γ) does have some impact on the demand for mortality-contingent claims, we find that the minimal amount of income required by the family is the

[5] We thank an anonymous JBF referee for pointing us in this direction, and the details on the modified HJB are available from the authors upon request

main driving factor, and not the risk tolerance of the family *per se*.

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Table #1: Life Insurance (in Thousands) and Allocation to Stocks at Age 45								
γ	c_d	c_l		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1.0$
1.5 (Low)	\$0	\$0	α_t^*	6.06	5.15	4.36	3.68	3.06
			I_t^*/λ_{y+t}	\$965	\$904	\$850	\$803	\$762
	\$10	\$12	α_t^*	4.54	3.62	2.82	2.14	1.54
			I_t^*/λ_{y+t}	\$917	\$854	\$800	\$754	\$714
	\$20	\$24	α_t^*	3.03	2.06	1.26	0.588	0.0226
			I_t^*/λ_{y+t}	\$869	\$802	\$746	\$700	\$666
3 (Medium)	\$0	\$0	α_t^*	3.63	2.82	2.15	1.57	1.05
			I_t^*/λ_{y+t}	\$965	\$898	\$842	\$797	\$762
	\$10	\$12	α_t^*	2.87	2.04	1.37	0.797	0.294
			I_t^*/λ_{y+t}	\$917	\$846	\$789	\$745	\$714
	\$20	\$24	α_t^*	2.12	1.25	0.579	0.0178	-0.465
			I_t^*/λ_{y+t}	\$869	\$787	\$727	\$684	\$666
4.5 (High)	\$0	\$0	α_t^*	2.82	2.05	1.41	0.873	0.384
			I_t^*/λ_{y+t}	\$965	\$892	\$835	\$792	\$762
	\$10	\$12	α_t^*	2.32	1.52	0.890	0.359	-0.122
			I_t^*/λ_{y+t}	\$917	\$838	\$779	\$738	\$714
	\$20	\$24	α_t^*	1.81	0.982	0.366	-0.157	-0.628
			I_t^*/λ_{y+t}	\$869	\$775	\$713	\$673	\$666
Parameter Assumptions: $\mu_m = 7\%, \sigma_m = 20\%, r = 2\%, \mu_w = 1\%, \sigma_w = 5\%$								
$M_0 = \$200, W_0 = \$50, T_R = 20, T_H = 50, \delta = 2\%, y = 45, m = 86.3, b = 9.5.$								

TABLE NOTES: The above table displays the results of our model for a middle-aged family whose breadwinner is 45 years old under various levels of risk aversion and minimal required (HARA) consumption levels. At higher levels of risk aversion (γ), under the same correlation (ρ) between shocks to wages and investment returns, the family invests less in risky equity, but holds relatively the same face value of life insurance.

Table #2: Life Insurance (in Thousands) and Allocation to Stocks at Age 55								
γ	c_d	c_l		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1.0$
1.5	\$0	\$0	α_t^*	2.15	1.96	1.78	1.61	1.46
(Low)			I_t^*/λ_{y+t}	\$485	\$469	\$455	\$442	\$430
	\$10	\$12	α_t^*	1.49	1.30	1.12	0.958	0.805
			I_t^*/λ_{y+t}	\$446	\$430	\$415	\$402	\$391
	\$20	\$24	α_t^*	0.837	0.636	0.458	0.297	0.150
			I_t^*/λ_{y+t}	\$407	\$389	\$374	\$362	\$352
3	\$0	\$0	α_t^*	1.22	1.05	0.886	0.736	0.596
(Medium)			I_t^*/λ_{y+t}	\$485	\$468	\$453	\$441	\$430
	\$10	\$12	α_t^*	0.897	0.717	0.556	0.407	0.268
			I_t^*/λ_{y+t}	\$446	\$428	\$413	\$400	\$391
	\$20	\$24	α_t^*	0.570	0.381	0.220	0.0749	-0.0592
			I_t^*/λ_{y+t}	\$407	\$385	\$369	\$357	\$352
4.5	\$0	\$0	α_t^*	0.917	0.745	0.589	0.445	0.308
(High)			I_t^*/λ_{y+t}	\$485	\$467	\$452	\$440	\$430
	\$10	\$12	α_t^*	0.699	0.523	0.368	0.225	0.0894
			I_t^*/λ_{y+t}	\$446	\$426	\$411	\$399	\$391
	\$20	\$24	α_t^*	0.481	0.296	0.142	0.0032	-0.129
			I_t^*/λ_{y+t}	\$407	\$382	\$365	\$354	\$352
Parameter Assumptions: $\mu_m = 7\%, \sigma_m = 20\%, r = 2\%, \mu_w = 1\%, \sigma_w = 5\%$								
$M_0 = \$400, W_0 = \$50, T_R = 10, T_H = 40, \delta = 2\%, y = 55, m = 86.3, b = 9.5$								

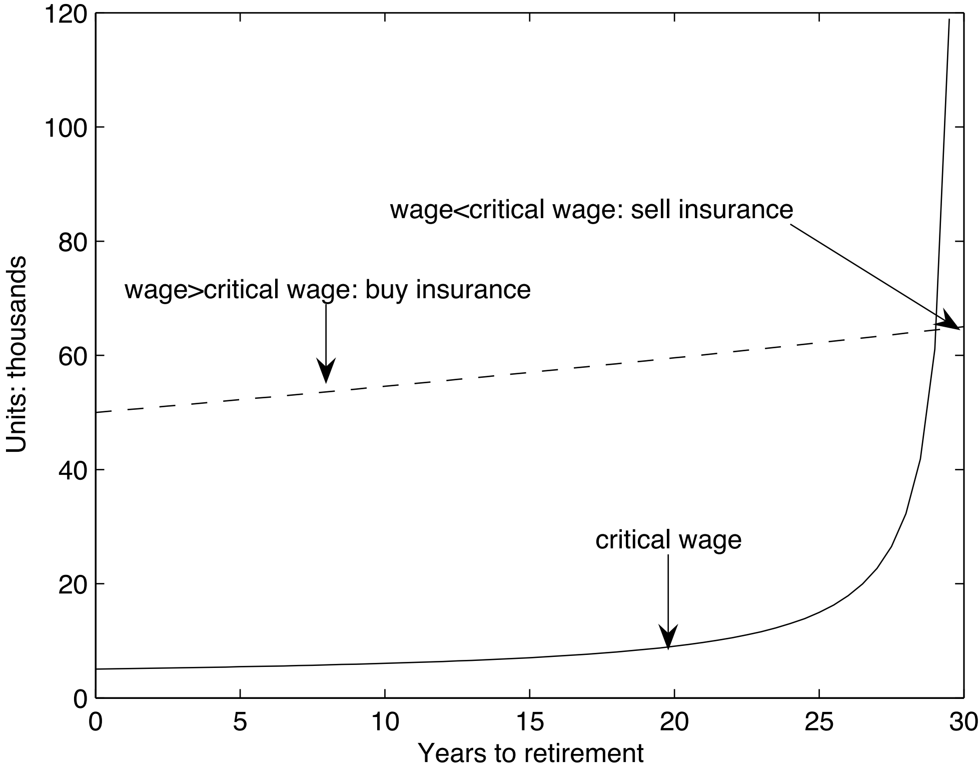
TABLE NOTES: The above table displays the results of our model for an older family (closer to retirement) whose breadwinner is 55 years old under various levels of risk aversion and minimal required (HARA) consumption levels. At higher levels of risk aversion (γ), under the same correlation (ρ) between shocks to wages and investment returns, the family invests less in risky equity, but holds relatively the same face value of life insurance.

Table #3: Life Insurance (in Thousands) and Allocation to Stocks at Age 35								
γ	c_d	c_l		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1.0$
1.5 (Low)	\$0	\$0	α_t^*	1.57K	1.25K	995	780	599
			I_t^*/λ_{y+t}	\$1.45K	\$1.32K	\$1.20K	\$1.11K	\$1.03K
	\$10	\$12	α_t^*	1.23K	911	651	438	262
			I_t^*/λ_{y+t}	\$1.39K	\$1.26K	\$1.14K	\$1.05K	\$970
	\$20	\$24	α_t^*	896	562	301	89.9	-75.7
			I_t^*/λ_{y+t}	\$1.34K	\$1.19K	\$1.08K	\$981	\$915
3 (Medium)	\$0	\$0	α_t^*	966	694	483	314	171
			I_t^*/λ_{y+t}	\$1.45K	\$1.30K	\$1.18K	\$1.09K	\$1.03K
	\$10	\$12	α_t^*	798	518	308	141	2.67
			I_t^*/λ_{y+t}	\$1.39K	\$1.24K	\$1.12K	\$1.03K	\$970
	\$20	\$24	α_t^*	629	340	135	-30.2	-166
			I_t^*/λ_{y+t}	\$1.34K	\$1.16K	\$1.04K	\$952	\$915
4.5 (High)	\$0	\$0	α_t^*	765	507	314	161	28.8
			I_t^*/λ_{y+t}	\$1.45K	\$1.29K	\$1.17K	\$1.08K	\$1.03K
	\$10	\$12	α_t^*	652	387	196	46.0	-83.7
			I_t^*/λ_{y+t}	\$1.39K	\$1.22K	\$1.09K	\$1.01K	\$970
	\$20	\$24	α_t^*	540	269	84.2	-66.3	-196
			I_t^*/λ_{y+t}	\$1.34K	\$1.14K	\$1.01K	\$932	\$915
Parameter Assumptions: $\mu_m = 7\%, \sigma_m = 20\%, r = 2\%, \mu_w = 1\%, \sigma_w = 5\%$								
$M_0 = \$1, W_0 = \$50, T_R = 30, T_H = 60, \delta = 2\%, y = 35, m = 86.3, b = 9.5$								

TABLE NOTES: The above table displays the results of our model for a young family whose breadwinner is 35 years old under various levels of risk aversion and minimal required (HARA) consumption levels. At higher levels of risk aversion (γ), under the same correlation (ρ) between shocks to wages and investment returns, the family invests less in risky equity, but holds relatively the same face value of life insurance.

Table #4: Life Insurance (in Thousands) and Allocation to Stocks at Age 45											
	ρ	-1.0		-0.5		0		0.5		1.0	
	f	0	15%	0	15%	0	15%	0	15%	0	15%
γ	$c_d = \$0,\ c_l = \$0$										
1.5	α_t^*	6.06	6.01	5.15	5.11	4.36	4.33	3.68	3.65	3.06	3.04
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$965	\$853	\$904	\$799	\$850	\$751	\$803	\$709	\$762	\$671
3	α_t^*	3.63	3.60	2.82	2.80	2.15	2.13	1.57	1.56	1.05	1.05
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$965	\$904	\$898	\$841	\$842	\$789	\$797	\$747	\$762	\$713
4.5	α_t^*	2.82	2.80	2.05	2.03	1.41	1.41	0.873	0.869	0.384	0.383
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$965	\$921	\$892	\$853	\$835	\$798	\$792	\$757	\$762	\$727
γ	$c_d = \$10,\ c_l = \$12$										
1.5	α_t^*	4.54	4.50	3.62	3.59	2.82	2.80	2.14	2.12	1.54	1.53
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$917	\$839	\$854	\$783	\$800	\$735	\$754	\$693	\$714	\$657
3	α_t^*	2.87	2.85	2.04	2.03	1.37	1.36	0.797	0.791	0.294	0.291
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$917	\$874	\$846	\$807	\$789	\$753	\$745	\$712	\$714	\$683
4.5	α_t^*	2.32	2.30	1.52	1.51	0.890	0.884	0.359	0.357	-0.122	-0.121
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$917	\$886	\$838	\$811	\$779	\$754	\$738	\$714	\$714	\$692
γ	$c_d = \$20,\ c_l = \$24$										
1.5	α_t^*	3.03	2.98	2.06	2.02	1.26	1.23	0.588	0.565	0.0226	0.005
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$869	\$822	\$802	\$762	\$746	\$712	\$700	\$671	\$666	\$640
3	α_t^*	2.12	2.09	1.25	1.23	0.579	0.565	0.0178	0.009	-0.465	-0.470
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$869	\$841	\$787	\$764	\$727	\$707	\$684	\$667	\$666	\$650
4.5	α_t^*	1.81	1.70	0.982	0.968	0.366	0.358	-0.157	-0.161	-0.628	-0.628
	$\frac{I_t^*}{\hat{\lambda}_{y+t}}$	\$869	\$847	\$775	\$758	\$713	\$698	\$673	\$660	\$666	\$653
Parameter Assumptions: $\mu_m = 7\%, \sigma_m = 20\%, r = 2\%, \mu_w = 1\%, \sigma_w = 5\%$											
$M_0 = \$200, W_0 = \$50, T_R = 20, T_H = 50, \delta = 2\%, y = 45, m = 86.3, b = 9.5.$											

TABLE NOTES: The above table displays the results of our model for a middle-aged family whose breadwinner is 45 years old under various levels of risk aversion and minimal required (HARA) consumption levels and two different loading factors.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100
80
60
40
20
0
Units: thousands
wage<critical wage: sell insurance
wage>critical wage: buy insurance
critical wage
0
5
10
15
20
25
30
Years to retirement