Optimal Asset Allocation and Ruin-Minimization Annuitization Strategies

Moshe A. Milevsky [1]

Kristen S. Moore [2]

Virginia R. Young [3]

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[1] Schulich School Of Business, York University, Toronto, Ontario, Canada

[2] Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA

Fax: 734 763 0937 e-mail: ksmoore@umich.edu phone: 734 615 6864

[3] Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA

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Abstract: In this paper, we derive the optimal investment and annuitization strategies for a retiree whose objective is to minimize the probability of lifetime ruin, namely the probability that a fixed consumption strategy will lead to zero wealth while the individual is still alive. Recent papers in the insurance economics literature have examined utility-maximizing annuitization strategies. Others in the probability, finance, and risk management literature have derived shortfall-minimizing investment and hedging strategies given a limited amount of initial capital. This paper brings the two strands of research together. Our model pre-supposes a retiree who does not currently have sufficient wealth to purchase a life annuity that will yield her exogenously desired fixed consumption level. She seeks the asset allocation and annuitization strategy that will minimize the probability of lifetime ruin. We demonstrate that because of the binary nature of the investor's goal, she will not annuitize any of her wealth until she can fully cover her desired consumption with a life annuity. We derive a variational inequality that governs the ruin probability and the optimal strategies, and we demonstrate that the problem can be recast as a related optimal stopping problem which yields a free-boundary problem that is more tractable. We numerically calculate the ruin probability and optimal strategies and examine how they change as we vary the mortality assumption and parameters of the financial model. Moreover, we solve the problem implicitly for the special case of exponential future lifetime. As a byproduct, we are able to quantify the reduction in lifetime ruin probability that comes from being able to manage the investment portfolio dynamically and purchase annuities.

JEL Classification: J26; G11

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1. Introduction and Motivation

Global pension reform and the trend towards privatization has focused much academic and practitioner attention on the market for voluntary life annuities, as an alternative to defined benefit pensions. While life annuities themselves are hundreds of years old – see Poterba (1997) for a brief history – it is only recently that they have attracted the attention of noted financial and insurance economists, such as Feldstein and Ranguelova (2001), as an alternative to Social Security.

In a well-cited paper from the public economics literature, Yaari (1965) proved that in the absence of annuity bequest motives – and in a deterministic financial economy – consumers will annuitize all of their liquid wealth. Richard (1975) generalized this result to a stochastic environment, and a recent paper by Davidoff, Brown, and Diamond (2003) demonstrates the robustness of the Yaari (1965) result. In practice, there are market imperfections, and frictions preclude full annuitization. Similarly, Brugiavini (1993), Kapur and Orszag (1999), Brown (2001), and Milevsky and Young (2003) provide theoretical and empirical guidance on the optimal time to annuitize under various market structures.

The common theme of the above-mentioned papers is the presumption of a rational utility-maximizing economic agent with rigid inter-temporal preferences and pre-specified relative risk aversion. While this von-Neumann-Morgenstern framework is the basis of most of micro-economic foundations, it is notably difficult to apply as a tool for normative advice.

Recently, though, a variety of papers in the risk and portfolio management literature have revitalized the Roy (1950) Safety-First rule and applied the concept to probability maximization of achieving certain investment goals. For example, Browne (1995, 1999a,b,c) derives the optimal dynamic strategy for a portfolio manager who is interested in minimizing the probability of shortfall. Within the insurance arena, Møller (2001) develops risk-minimizing hedging strategies for savings policies with random death benefits.

Indeed, there is something intuitively appealing about minimizing the probability of shortfall that lends itself to asset allocation advice. In fact, in the U.S., the Nobel laureate William Sharpe has founded a financial services advisory firm that is largely based on using probabilities to provide investment advice.

Therefore, motivated by the desire to apply the probability optimization concept to the retirement and annuity literature, in this paper we find the optimal annuity-purchasing scheme for an individual who seeks to minimize the probability that she outlives her wealth, also called the probability of lifetime ruin. In other words, we assume the retiree will maintain a pre-specified (exogenous) consumption level, and we provide guidance on the optimal investment strategy, as well as the optimal time to annuitize, in order to minimize the probability that wealth will reach zero while the individual is still alive.

Milevsky and Robinson (2000) introduced the probability of lifetime ruin as a risk-metric for retirees, albeit in a static environment. Similarly, Young (2004) determines the optimal investment policy for an individual who consumes at a specific rate, who invests in a complete financial market, and who does not buy annuities. By contrast, we allow the individual to buy annuities, as well as to invest in a financial market. The irreversibility of annuity purchases and their illiquidity creates a complex optimization environment, which renders many classical results inoperable. Of course, these same challenges are what make the problem mathematically interesting.

Our agenda for this paper is as follows. In Section 2, we introduce the concept of self-annuitization and provide some general statements about the probability of lifetime ruin under such a strategy. We, then, present our formal optimization model and and use optimal stochastic control to derive a variational inequality that governs the ruin probability and optimal strategies. We show that the annuitization strategy is a barrier strategy defined by the barrier at which the marginal ruin probability with respect to annuity income and the (adjusted) marginal ruin probability with respect to wealth are equal. This type of result – namely, taking no action until the marginal benefit is at least equal to the marginal cost – is seen often in the economics literature. The annuity-purchasing problem is qualitatively similar to the problem of optimal consumption and investment in the presence of proportional transaction costs. The difference between the two problems is that for us, once the individual's wealth reaches the barrier, then she annuitizes all her wealth, and the "game" is over, as we show in Section 3. Friedman and Shen (2002) applied similar stochastic control methods to problems in retirement planning and insurance.

In Section 3, we reduce the dimension of the variational inequality obtained in Section 2. In Section 2, the probability of lifetime ruin is given as a function of the current time, the wealth w at that time, and the annuity income A at that time. If c denotes the desired consumption rate, then it turns out the probability of lifetime ruin is a function of $z = w/(c - A)$ and time, so we can reduce the dimension of the problem by one. We, then, study properties of the optimal investment and annuity-purchasing policies. We show that if the wealth-to-desired additional consumption (or desired consumption minus income) ratio is greater than or equal to the actuarial present value of a continuous annuity that pays \$1 per year, then the individual will purchase a lump sum annuity to guarantee her desired consumption rate so that she will never ruin. Conversely, if the wealth-to-(consumption minus income) ratio is less than the actuarial present value of the continuous annuity, then the individual will buy no annuity at that time but wait until wealth is great enough, a rather surprising bang-bang result that is inherited from the nature of the investor's goal and stands in contrast to the instantaneous control policy that would apply if the objective were to maximize expected utility of lifetime consumption and bequest (Milevsky and Young, 2003). In Section 4, we use duality techniques to transform the nonlinear partial differential equation for the probability of lifetime ruin (with known boundary conditions) to a linear free-boundary problem. In Section 5, we solve the free-boundary problem for a special case and contrast our results with those in Section 2.1, where no annuities or risky assets were available to the individual. In this way, we quantify the benefits of dynamic portfolio management, in which the portfolio includes life annuities and a risky asset. In Section 6, we use the connection between free-boundary problems and variational inequalities for optimal stopping problems in order to compute ruin probabilities and optimal strategies for more general cases than the one considered in Section 5. Section 7 concludes the paper.

2. Probability of Lifetime Ruin

In this section, we formulate our models for the probability of lifetime ruin–first in the case of deterministic returns, then in the case of stochastic returns.

2.1. Self-Annuitization with Deterministic Returns

In this section, we consider a simple model in which an individual can invest only in a risk-free asset earning rate r. We assume that she begins with wealth 1 and self-annuitizes; that is, she consumes a level amount c per year until she dies or runs out of money, whichever comes first. We compute the time of ruin, and under the assumption of exponential future lifetime, we compute the probability of lifetime ruin. In Section 5.2, we contrast these results with those for an investor who can trade dynamically between risk-free and risky assets and who can purchase annuities instead of self-annuitizing.

We start with a future lifetime random variable τ_d that is exponentially distributed, for which the probability of survival is given by

$$\Pr[\tau_d > t] = e^{-\lambda t},$$

in which λ is the instantaneous hazard rate (or force of mortality). The greater the hazard rate λ, the lower the probability of survival to any given age t.

Under an exponential mortality assumption, the expected (or mean) future lifetime is equal to

$$E[\tau_d] = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}.$$

Thus, for example, if $\lambda = 0.05$, then the life expectancy is 20 years, while if $\lambda = 0.1$, the life expectancy is 10 years. Note, however, that the median future lifetime – which is distinct from the mean future lifetime – is the value m of t at which the probability of survival is exactly equal to 50%. Both median and mean lifetime are valid measures of central tendency for human mortality, and both are used in daily practice (unfortunately with much confusion at times). Thus, inverting the survival probability equation to obtain the 50% mark leads to:

$$0.5 = e^{-\lambda m} \text{ implies } m = \frac{\ln 2}{\lambda} < \frac{1}{\lambda}.$$

For example, when $\lambda = 0.05$, the median future lifetime $m = \ln 2/0.05 = 13.862$ years, and when $\lambda = 0.1$, we get that $m = \ln 2/0.1 = 6.931$ years. Note that m is always less than life expectancy, as evidenced by the above inequality. The distinction between median and mean lifetime is critical for understanding lifetime ruin probabilities. Recall that 50% of the current population cohort is dead at the median lifetime point, while the other 50% survives. However, the probability of surviving to the mean lifetime of $1/\lambda$ is a much lower $e^{-\lambda/\lambda} = e^{-1} = 36.78\%$. The skewness or asymmetry in the future lifetime random variable can be measured by the gap between the mean and median lifetime, which is $(1 - \ln 2)/\lambda$. The lower the hazard rate, the greater the asymmetry. In this paper, we refer to life annuity prices. Formally, the price (or present value) of a fixed \$1 per annum payout annuity (for life, with no guarantee period) that is paid continuously is computed via

$$\int_0^\infty e^{-rt} \Pr[\tau_d > t] dt.$$

It is effectively equal to the present value of \$1 per annum discounted by the risk-free rate and the probability of survival. The greater the interest rate, the lower the present value of the life annuity. In the event of exponential mortality with hazard rate λ, the annuity price equals

$$\int_0^\infty e^{-rt} e^{-\lambda t} dt = \frac{1}{r + \lambda}.$$

Thus, for example, if the hazard rate is $\lambda = 0.05$ (which means that future life expectancy is 20 years) and the interest rate in the (annuity) market is $r = 0.07$, the price of \$1 for life is $1/0.12 = 8.33$ dollars. Stated differently, one dollar of initial premium will yield a fixed annuity payout of $\lambda + r = 0.12$ dollars per annum for life. However, when the (initial) future life expectancy is only 10 years, which implies the hazard rate is $\lambda = 0.10$, then under an $r = 0.07$ interest rate, the cost of \$1 for life is only $1/0.17 = 5.88$ dollars. Equivalently, the payout per initial dollar of premium is 0.17 dollars per annum.

In the deterministic case, we use the function $W(t)$ to denote the wealth at time t of the retiree assuming she does not annuitize. Instead, she consumes a constant amount c per annum until she either runs out of money or she dies (whichever comes first). We quantify the dynamics of the wealth process and compute the probability she will run out of money while she is still alive. To simplify our work, we assume only one interest rate r (i.e., no term structure or expenses) in the economy and that all annuities are priced (fairly) as a function of the hazard rate only (i.e., we ignore loading and expenses).

Formally, under a self-annuitization strategy, the wealth process of the retiree will obey the ordinary differential equation:

$$dW(t) = (rW(t) - c)dt, \quad W(0) = 1.$$

The individual retires with \$1 of wealth, invests at a rate of r and consumes and a rate of c. Intuitively, therefore, wealth increases at the interest rate at which money is invested minus the consumption rate. The solution to this ordinary differential equation is

$$W(t) = e^{rt} - c\left(\frac{e^{rt} - 1}{r}\right), \quad t \leq t^*,$$

and 0 after time t^*, in which t^* is the point at which the process hits zero (i.e., the individual is ruined). One can interpret this expression for $W(t)$ as the value at time t of the initial \$1 minus the accumulated value of the continuous withdrawal due to consumption at rate c.

Now, assume the consumption rate is set equal to exactly $c = \lambda + r$, which is the amount of life annuity income that \$1 will provide. In this case, the function for wealth is

$$W(t) = 1 - \frac{\lambda}{r}(e^{rt} - 1), \quad t \leq t^*.$$

More importantly, the rate at which wealth evolves, the derivative of the wealth function, is $-\lambda e^{rt}$, which is always negative, and more so the greater the value of r. In other words, when $c = \lambda + r$, the greater the interest rate, the higher is the rate at which wealth is depleted under a self-annuitization strategy. This is critical for understanding why the interest rate has such a strong impact on the ruin time, even though the life expectancy (or hazard rate) does not change.

Finally, in this self-annuitization case, the ruin time t^*, the point at which the function $W(t)$ reaches zero, can be simplified to

$$t^* = \frac{1}{r}\ln\left(1 + \frac{r}{\lambda}\right).$$

Note that the derivative of t^* with respect to r is negative; that is, as the interest rate increases, the time of ruin decreases.

For example, if $r = 0.07$ and $\lambda = 0.05$ (a life expectancy of 20 years), then the ruin time is $t^* = \ln(1 + 0.07/0.05)/0.07 = 12.51$ years. However, if the interest rate is exactly $r = 0.05$, then the ruin time is $t^* = \ln(1 + 0.05/0.05)/0.05 = 13.86$ years. In words, when the pricing interest rate is reduced – and the replicating consumption strategy is reduced accordingly – the ruin time is later. Note that when $r = \lambda$, the value of $t^* = \ln 2/\lambda$, which is exactly the median life span. In other words, when the interest rate is equal to the hazard rate, the money runs out at median life expectancy. Finally, in the limit, as r goes to zero, the ruin time is precisely the life expectancy $1/\lambda$ because

$$\lim_{r \to 0^+} \frac{1}{r} \ln\left(1 + \frac{r}{\lambda}\right) = \frac{1}{\lambda}.$$

Note that the money runs out earlier than life expectancy when the interest rate is positive because $c = \lambda + r$. Combining the survival equation and the ruin equation, the probability of surviving to the point at which the funds are exactly exhausted is

$$\Pr[\tau_d > t] = \exp(-\lambda t^*) = \left(1 + \frac{r}{\lambda}\right)^{-\frac{\lambda}{r}}.$$

First, when $\lambda = r$, which means that the hazard rate is exactly equal to the pricing interest rate, the probability of ruin (i.e., being alive when the money runs out) is exactly equal to 50%. In fact, this is the only case for which it is equal to 50%. Furthermore, when the interest rate is higher than the hazard rate, the probability of lifetime ruin is greater than 50%, and when the interest rate is lower than the hazard rate, the probability of lifetime ruin is lower than 50%. Note also that the survival probability decreases with λ, increases with r, and converges to e^{-1} as $\lambda \to \infty$ or as $r \to 0$. In Section 5.2, we contrast this deterministic-return case with one for which we allow random returns.

2.2. Stochastic Returns

In this subsection, we formalize the optimal annuity-purchasing and optimal investment problem for an individual who seeks to minimize the probability that she outlives her wealth. A priori, we allow the individual to buy annuities in lump sums or continuously, whichever is optimal. Our results are similar to those of Dixit and Pindyck (1994, pp 359ff), which are given in the context of real options. They consider the problem of a firms (irreversible) capacity expansion. In our model, annuity purchases are also irreversible, and this leads to the similarity in results.

We assume that the individual can invest in a riskless asset whose price at time s, X_s, follows the process $dX_s = rX_s ds, X_t = x > 0$, for some fixed $r \geq 0$, as in the previous subsection. However, unlike the previous subsection, the individual can also invest in a risky asset whose price at time s, S_s, follows geometric Brownian motion given by

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dB_s, \\ S_t = S > 0, \end{cases}$$

in which $\mu > r, \sigma > 0$, and B_s is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_s\}$ of the probability space $(\Omega, \mathcal{F}, \Pr)$. Let W_s be the wealth at time s of the individual (after possibly purchasing annuities at that time), and let π_s be the amount that the decision maker invests in the risky asset at time s. It follows that the amount invested in the riskless asset is $W_s - \pi_s$. Also, the decision maker consumes at a constant rate of c.

Our economy can be either real or nominal. When our model is interpreted in nominal terms, then the consumption rate c is nominal, and we assume that the individual buys annuities that pay a fixed nominal amount. In practice, of course, this exposes the retiree to inflation risk since c today will buy much more than c in twenty years. However, if c is real, then we assume that the individual (only) has access to annuities that are indexed to inflation and thereby pay a fixed real amount. Also, in this case, the returns on the riskless and risky assets are stated in real returns. We prefer to think of the model in real terms, and our numerical examples are presented in real terms so that inflation risk, which would be a problem if c were stated in nominal terms, is not an issue.

We employ standard actuarial notation as in Bowers et al. (1997). Let ${}_tp_x^S$ denote the individual-specific conditional probability that an individual aged (x) will survive to age $(x+t)$. It is defined via the individual-specific hazard function, λ_{x+s}^S, by the formula ${}_tp_x^S = \exp\left(-\int_0^t \lambda_{x+s}^S ds\right)$. We have a similar formula for the conditional probability of survival used in pricing annuities, ${}_tp_x^O$, in terms of the pricing (or objective) hazard function, λ_{x+s}^O. The actuarial present value of a life annuity that pays \$1 per year continuously to (x) is written $\bar{a}_x^O$. It is defined by $\bar{a}_x^O = \int_0^\infty e^{-rt} {}_tp_x^O dt$. Note that these formulas all generalize those of the previous subsection.

Just to clarify, by $\bar{a}_x^O$, we mean the actual market price of the life annuity. We deliberately refrain from discussing anti-selection, which creates the wedge between individual-specific and pricing (or objective) hazard rates. In addition, we omit actuarial loading fees, agent commissions, and other market imperfections that only add to the cost of annuities and can be absorbed in the pricing hazard rate.

The individual has a non-negative (annuity) income rate at time s of A_s after any annuity purchases at that time (A_{s-} before any annuity purchases at time s). The exogenous initial income could include Social Security benefits and defined benefit pension benefits, for example. We assume that she can purchase an annuity at the (unloaded) price of $\bar{a}^O_{x+s}$ per dollar of annuity income at time s, or equivalently, at age $x+s$. Thus, wealth follows the process

$$\begin{cases} dW_s = [rW_{s-} + (\mu - r)\pi_{s-} + A_{s-} - c]\, ds + \sigma \pi_{s-} dB_s - \bar{a}^O_{x+s} dA_s, \\ W_t = w \geq 0. \end{cases} \tag{2.1}$$

The negative sign for the subscript on the random processes denotes the left-hand limit of those quantities before any (lump-sum) annuity purchases.

We assume that the decision maker seeks to minimize, over admissible $\{\pi_s, A_s\}$, her probability of lifetime ruin, namely, the probability that her wealth drops to zero before she dies. Admissible $\{\pi_s, A_s\}$ are those that are measurable with respect to the information available at time s, namely $\mathcal{F}_s$, that restrict the annuity-income process to be non-negative and non-decreasing (i.e., annuity purchases are irreversible), and that result in (2.1) having a unique solution; see Karatzas and Shreve (1998), for example. Note that π_s is unconstrained; thus, the investment in the risky asset can exceed current wealth (and often does, as we will see in Section 6.3). The individual values her probability of lifetime ruin via her specific hazard rate (or force of mortality), while annuities are priced by using the objective hazard rate.

Denote the random time of death of our individual by τ_d, as in the previous subsection, and the random time of lifetime ruin by τ_0; that is, τ_0 is the time at which wealth reaches zero. Thus, the probability of lifetime ruin ψ for the individual at time t, or age $x+t$, defined on $\bar{D} = \{(w, A, t) : 0 \leq w \leq (c - A)\bar{a}^O_{x+t}, 0 \leq A \leq c, t \geq 0\}$ is given by

$$\psi(w, A, t) = \inf_{\{\pi_s, A_s\}} \Pr\left[\tau_0 < \tau_d | W_t = w, A_t = A\right]. \tag{2.2}$$

Note that if $w \geq (c - A)\bar{a}^O_{x+t}$, then the individual can purchase an annuity that will guarantee her an income of $(c - A)$, which added to her income of A, gives her income to match her consumption rate of c. Thus, $\psi(w, A, t) = 0$ for $w \geq (c - A)\bar{a}^O_{x+t}$. If life annuities were not available, securing lifetime income would necessitate acquiring a perpetuity, which would cost $1/r$ per \$1 of income, much more than the annuity. This was the problem analyzed by Young (2004).

We continue with a formal derivation of the associated Hamilton-Jacobi-Bellman (HJB) variational inequality. Suppose that at the point (w, A, t), it is optimal *not* to purchase any annuities. It follows from Itô's lemma that ψ satisfies the equation

$$\lambda^S_{x+t}\psi = \psi_t + (rw + A - c)\psi_w + \min_\pi \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww}\right]. \tag{2.3}$$

Because the above policy is in general suboptimal, (2.3) holds as an inequality; that is, for all (w, A, t),

$$\lambda^S_{x+t}\psi \le \psi_t + (rw + A - c)\psi_w + \min_\pi \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right]. \tag{2.4}$$

Next, assume that at the point (w, A, t), it is optimal to buy an annuity instantaneously. In other words, assume that the investor moves instantly from (w, A, t) to $(w - \bar{a}^O_{x+t}\Delta A, A + \Delta A, t)$. Then, the optimality of this decision implies that

$$\psi(w, A, t) = \psi(w - \bar{a}^O_{x+t}\Delta A, A + \Delta A, t), \tag{2.5}$$

which in turns yields

$$\bar{a}^O_{x+t}\psi_w(w, A, t) - \psi_A(w, A, t) = 0. \tag{2.6}$$

Note that the lump-sum purchase is such that the derivative of the probability of lifetime ruin with respect to annuity income equals the adjusted derivative with respect to wealth, in which we adjust by the cost of \$1 of annuity income $\bar{a}^O_{x+t}$. This is parallel to many results in economics. Indeed, the derivative of the probability of lifetime ruin with respect to annuity income can be thought of as (the negative of) the marginal utility of the benefit, while the adjusted derivative with respect to wealth can be thought of as (the negative of) the marginal utility of the cost. We say 'negative' here because ψ is decreasing with respect to w and A. Thus, the lump-sum purchase forces the marginal utilities of benefit and cost to equal.

However, such a lump-sum purchasing policy is in general suboptimal; therefore, (2.6) holds as an inequality and becomes

$$\bar{a}^O_{x+t}\psi_w(w, A, t) - \psi_A(w, A, t) \le 0. \tag{2.7}$$

By combining (2.4) and (2.7), we obtain the HJB variational inequality (2.8) below associated with the probability of ruin ψ given in (2.2). The following result can be proved as in Zariphopoulou (1992), for example.

Proposition 2.1: *The probability of lifetime ruin is a constrained viscosity solution of the Hamilton-Jacobi-Bellman variational inequality*

$$\max \left[\lambda^S_{x+t}\psi - \psi_t - (rw + A - c)\psi_w - \min_\pi \left((\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right), \bar{a}^O_{x+t}\psi_w - \psi_A \right] = 0. \tag{2.8}$$

In the next section, we show that the barrier in (2.6) is the line $w = (c - A)\bar{a}^O_{x+t}$; thus, the individual will annuitize when she has sufficient wealth to cover her shortfall of $c - A$. If wealth and annuity income initially lie to the right of the barrier at time t, i.e., $w > (c - A)\bar{a}^O_{x+t}$, then the individual will immediately spend a lump sum of wealth to guarantee that the probability of lifetime ruin is zero. See Figure 2.1. Otherwise, the annuity income is constant when wealth is low enough, i.e., $w < (c - A)\bar{a}^O_{x+t}$. Once wealth is high enough, i.e., $w = (c - A)\bar{a}^O_{x+t}$, the individual will spend her wealth to guarantee an income rate of $(c - A) + A = c$ to match her consumption rate of c.

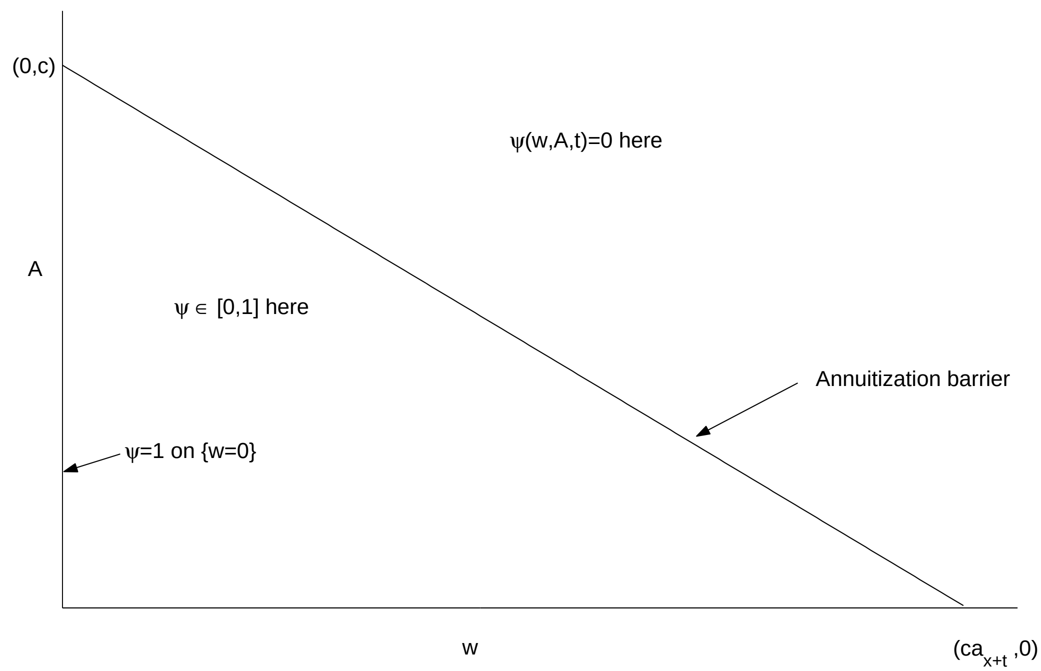

Figure 2.1: Wealth-annuity space and the annuitization barrier for fixed time

Thus, as in Dixit and Pindyck (1994, pp 359ff) or in Zariphopoulou (1992), we have discovered that the optimal annuity-purchasing scheme is a type of barrier control. Other barrier control policies appear in finance and insurance. In finance, Zariphopoulou (1999, 2001) reviews the role of barrier policies in optimal investment in the presence of transaction costs; also see the references within her two articles. See Gerber (1979) for a classic text on risk theory in which he includes a section on optimal dividend payout and shows that it follows a type of barrier control. See Neuberger (2002) for an analysis that is similar to ours.

3. Reducing the Dimension of the Minimization Problem

In this section, we show that we can reduce the dimension of the variational inequality (2.8) by transforming the ruin probability $\psi(w, A, t)$ to a function of two variables. We also show that the barrier described by (2.6) corresponds with the line $w = (c - A)\bar{a}^O_{x+t}$.

The probability of lifetime ruin ψ is a function of the ratio $z = w/(c - A)$ and time t. This observation is also made in Milevsky and Robinson (2000), where the probability of lifetime ruin is shown to depend only on the ratio of current wealth to desired consumption. The easiest way to see this homogeneity is to define $\tilde{\psi}(w, \tilde{A}, t) = \psi(w, c - \tilde{A}, t)$. Then, $\psi(w, A, t) = \tilde{\psi}(w, c - A, t) = \tilde{\psi}(z(c - A), c - A, t) = \tilde{\psi}(z, 1, t)$ with targeted consumption $c/(c - A)$, in which the last equality follows from scaling the entire problem by $(c - A)$. Thus, define V by

$$V(z, t) = \tilde{\psi}(z, 1, t), \tag{3.1}$$

so that $\psi(w, A, t) = V(z, t)$, $\psi_t = V_t$, $\psi_w = \frac{1}{c-A} V_z$, $\psi_{ww} = \left(\frac{1}{c-A}\right)^2 V_{zz}$, and $\psi_A = \frac{z}{c-A} V_z$. Then, the barrier equation in (2.6) becomes

$$z V_z = \bar{a}^O_{x+t} V_z;$$

thus, either $V_z = 0$ at the barrier or $z = \bar{a}^O_{x+t}$ there. If we assume that $V_z = 0$, we obtain a contradiction, and we omit the proof of this. Therefore, we have $z = \bar{a}^O_{x+t}$ at the barrier and $z < \bar{a}^O_{x+t}$ in the region for which annuity buying is not optimal.

We have just shown that the individual will buy *no* annuities unless $w \geq (c - A)\bar{a}^O_{x+t}$, in which case the individual will spend at least $(c - A)\bar{a}^O_{x+t}$ to buy a life annuity to guarantee income of $(c - A)$ from to the annuity. This income plus the income A covers the consumption rate c, and the individual will not ruin. Therefore, the individual will not buy an annuity until she can guarantee that she will not ruin, a type of "bang-bang" strategy that results from the all-or-nothing nature of her goal.

From the preceding discussion, we have the following proposition.

Proposition 3.1: *For each value of $t \geq 0$,*

(i) If $z \geq \bar{a}^O_{x+t}$, then the individual immediately buys an annuity to guarantee total income of at least c;

(ii) If $z < \bar{a}^O_{x+t}$, then the individual buys no annuity; i.e., she is in the region of inaction.

It follows that at each time point, the barrier is a ray emanating from the origin and lying in the first quadrant of $(w, c - A)$ space. Equivalently, in (w, A) space, the barrier is the line segment in the first quadrant with equation $A = c - w/\bar{a}^O_{x+t}$, as in Figure 2.1.

Davis and Norman (1990) and Shreve and Soner (1994) find a similar result for the problem of optimal consumption and investment in the presence of proportional transaction costs.

It follows that the HJB variational inequality for ψ from Proposition 2.1 becomes the following one for V on $0 < z \le \bar{a}^O_{x+t}$:

$$\max\left[\lambda^S_{x+t} V - V_t - (rz-1)V_z - \min_{\hat{\pi}}\left((\mu - r)\hat{\pi} V_z + \frac{1}{2}\sigma^2 \hat{\pi}^2 V_{zz}\right), z - \bar{a}^O_{x+t}\right] = 0, \quad (3.2)$$

in which $\hat{\pi} = \frac{\pi}{c-A}$. Davis and Norman (1990) and Shreve and Soner (1994) use a similar transformation in the problem of consumption and investment in the presence of transaction costs. Also, Duffie et al. (1997) and Koo (1998) use a similar transformation to study optimal consumption and investment with stochastic income.

We are now ready to give a complete formulation of the probability of lifetime ruin ψ.

Proposition 3.2: *The probability of lifetime ruin ψ in (2.2) is given by*

$$\psi(w, A, t) = V(z, t) \text{ if } z := w/(c-A) < \bar{a}^O_{x+t}; \text{ otherwise, } \psi(w, A, t) = 0,$$

in which V solves

$$\lambda^S_{x+t} V = V_t + (rz-1)V_z + \min_{\hat{\pi}}\left((\mu - r)\hat{\pi} V_z + \frac{1}{2}\sigma^2 \hat{\pi}^2 V_{zz}\right), \quad (3.3)$$

for $z < \bar{a}^O_{x+t}$, with boundary conditions $V(0,t) = 1$ and $V(\bar{a}^O_{x+t}, t) = 0$ and with transversality condition $\lim_{s\to\infty} {}_{s-t}p^S_{x+t} E[V(Z^*_s, s) | Z_t = z] = 0$, in which Z^*_s is the optimally controlled Z_s.

4. Linearizing the Equation for V via Duality Arguments

In this section, we transform the nonlinear boundary-value problem in (3.3) to a linear free-boundary problem. To this end, we first eliminate the $\lambda^S_{x+t} V$ term from (3.3) by defining

$$f(z, t) = {}_t p^S_x V(z, t).$$

It follows that (3.3) becomes

$$f_t + (rz-1) f_z + \min_{\hat{\pi}}\left[(\mu - r)\hat{\pi} f_z + \frac{1}{2}\sigma^2 \hat{\pi}^2 f_{zz}\right] = 0, \quad (4.1)$$

with boundary conditions $f(0,t) = {}_t p^S_x$ and $f(\bar{a}^O_{x+t}, t) = 0$ and with transversality condition $\lim_{s\to\infty} E[f(Z^*_s, s) | Z_t = z] = 0$. This condition can be rewritten as $\lim_{t\to\infty} f(z,t) = 0$ with probability 1 because $0 \le f \le 1$.

Next, consider the concave dual of f defined by

$$\tilde{f}(y,t) = \min_{z>0}[f(z,t) + zy]. \tag{4.2}$$

The critical value z^* solves the equation $f_z(z,t) + y = 0$; thus, $z^* = I(-y,t)$, in which I is the inverse of f_z with respect to z. It follows that

$$\tilde{f}(y,t) = f[I(-y,t),t] + yI(-y,t). \tag{4.3}$$

Note that

$$\begin{aligned} \tilde{f}_y(y,t) &= -f_z[I(-y,t)]I_y(-y,t) + I(-y,t) - yI_y(-y,t) \\ &= yI_y(-y,t) + I(-y,t) - yI_y(-y,t) \\ &= I(-y,t). \end{aligned} \tag{4.4}$$

We can retrieve the function f from $\tilde{f}$ by the relationship

$$f(z,t) = \max_{y>0}[\tilde{f}(y,t) - zy]. \tag{4.5}$$

Indeed, the critical value y^* solves the equation $\tilde{f}_y(y,t) - z = 0$; thus, $y^* = -f_z(z,t)$, and

$$\begin{aligned} \tilde{f}(y^*,t) - zy^* &= f[I(-y^*,t),t] + y^*I(-y^*,t) - zy^* \\ &= f[I(f_z(z,t),t),t] - f_z(z,t)I(f_z(z,t),t) + zf_z(z,t) \\ &= f(z,t) - zf_z(z,t) + zf_z(z,t) \\ &= f(z,t), \end{aligned}$$

in which we use equation (4.3) for the first equality.

Next, note that

$$\tilde{f}_{yy}(y,t) = -I_y(-y,t) = -1/f_{zz}[I(-y,t),t], \tag{4.6}$$

and

$$\begin{aligned} \tilde{f}_t(y,t) &= f_z[I(-y,t),t]I_t(-y,t) + f_t[I(-y,t),t] + yI_t(-y,t) \\ &= -yI_t(-y,t) + V_t[I(-y,t),t] + yI_t(-y,t) \\ &= f_t[I(-y,t),t]. \end{aligned} \tag{4.7}$$

In the partial differential equation for f, let $z = I(-y,t)$ to obtain

$$f_t[I(-y,t),t] + (rI(-y,t) - 1)f_z[I(-y,t),t] - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(f_z[I(-y,t),t])^2}{f_{zz}[I(-y,t),t]} = 0.$$

Rewrite this equation in terms of $\tilde{f}$ to get

$$\tilde{f}_t(y,t) + (rI(-y,t) - 1)(-y) - m\frac{(-y)^2}{-1/\tilde{f}_{yy}(y,t)} = 0,$$

in which $m = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$, or equivalently,

$$\tilde{f}_t(y,t) - ry\tilde{f}_y(y,t) + my^2\tilde{f}_{yy}(y,t) + y = 0, \tag{4.8}$$

with boundary conditions given implicitly by $f(0,t) = {}_tp_x^S$ and $f(\bar{a}_{x+t}^O, t) = 0$. Note that (4.8) is a *linear* partial differential equation.

Now, consider the boundary conditions $f(0,t) = {}_tp_x^S$ and $f(\bar{a}_{x+t}^O, t) = 0$. Because $f_z < 0$ is strictly increasing with respect to z, we have $y_0(t) > y_b(t) \geq 0$ for all $t \geq 0$, in which $y_0(t)$ and $y_b(t)$ are defined by

$$y_0(t) = -f_z(0,t), \tag{4.9}$$

and

$$y_b(t) = -f_z(\bar{a}_{x+t}^O, t). \tag{4.10}$$

We use subscript 0 to denote the point corresponding to wealth equal to 0, and we use subscript b to denote the point corresponding to the value of wealth at which the individual **b**uys a life annuity.

Thus, the boundary conditions become

$$\tilde{f}(y_0(t), t) = {}_tp_x^S, \text{ for } \tilde{f}_y(y_0(t), t) = 0, \tag{4.11}$$

and

$$\tilde{f}(y_b(t), t) = \bar{a}_{x+t}^O y_b(t), \text{ for } \tilde{f}_y(y_b(t), t) = \bar{a}_{x+t}^O. \tag{4.12}$$

The transversality condition $\lim_{t\to\infty} f(z,t) = 0$ with probability 1 becomes $\lim_{t\to\infty} \tilde{f}(y,t) = 0$ with probability 1. Note that the first equations in (4.11) and (4.12) are reminiscent of value matching conditions, while the second equations are reminiscent of smooth pasting conditions. We exploit this observation in Section 6, where we express $\tilde{f}$ as the value function for an optimal stopping problem. Thus, we are able to solve for $\tilde{f}$ via a numerical method such as projected SOR (Wilmott, Dewynne, and Howison, 2000). Before pursuing this numerical method, we obtain an exact solution for ψ in a special case in Section 5.

5. Solution of the Free-Boundary Problem for a Special Case: Constant Force of Mortality

Throughout this section, we assume that the forces of mortality are constant; i.e., that $\lambda_{x+t}^S \equiv \lambda^S$ and $\lambda_{x+t}^O \equiv \lambda^O$ for all $t \geq 0$. In this case, the ruin probability ψ is independent of time, and the partial differential equation in (3.3) is an ordinary differential equation. We use this time-homogeneity to compute a "implicit" analytical solution of ψ.

5.1. Solution of the Boundary-Value Problem

If we assume that the forces of mortality are constant, (3.3) becomes the ordinary differential equation:

$$\lambda_{x+t}^S V = (rz - 1)V' + \min_{\hat{\pi}} \left((\mu - r)\hat{\pi} V' + \frac{1}{2}\sigma^2 \hat{\pi}^2 V'' \right), \tag{5.1}$$

with boundary conditions $V(0) = 1$ and $V(1/(r + \lambda^O)) = 0$.

If we define the dual of V by $\tilde{V}(n) = \min_{z>0}[V(z) + zn]$, as in Section 4, then we obtain the following free-boundary problem for $\tilde{V}$:

$$-\lambda^S \tilde{V}(n) - (r - \lambda^S) y \tilde{V}'(n) + m y^2 \tilde{V}''(n) + y = 0, \tag{5.2}$$

with boundary conditions

$$\tilde{V}(n_0) = 1, \text{ for } \tilde{V}'(n_0) = 0, \tag{5.3}$$

and

$$\tilde{V}(n_b) = \frac{n_b}{r + \lambda^O}, \text{ for } \tilde{V}'(n_b) = \frac{1}{r + \lambda^O}. \tag{5.4}$$

The general solution of (5.2) is

$$\tilde{V}(n) = D_1 n^{B_1} + D_2 n^{B_2} + \frac{n}{r}, \tag{5.5}$$

with D_1 and D_2 constants determined by the boundary conditions, and with B_1 and B_2 given by

$$B_1 = \frac{1}{2m} \left[(r - \lambda^S + m) + \sqrt{(r - \lambda^S + m)^2 + 4m\lambda^S} \right] > 1, \tag{5.6}$$

and

$$B_1 = \frac{1}{2m} \left[(r - \lambda^S + m) - \sqrt{(r - \lambda^S + m)^2 + 4m\lambda^S} \right] < 0. \tag{5.7}$$

The boundary conditions at n_b give us

$$D_1 n_b^{B_1} + D_2 n_b^{B_2} + \frac{n_b}{r} = \frac{n_b}{r + \lambda^O}, \tag{5.8}$$

and

$$D_1 B_1 n_b^{B_1} + D_2 B_2 n_b^{B_2} + \frac{n_b}{r} = \frac{n_b}{r + \lambda^O}. \tag{5.9}$$

Solve equations (5.8) and (5.9) to get D_1 and D_2 in terms of n_b:

$$D_1 = -\frac{\lambda^O}{r(r+\lambda^O)} \frac{1 - B_2}{B_1 - B_2} n_b^{1-B_1} < 0, \tag{5.10}$$

and

$$D_2 = -\frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1 - 1}{B_1 - B_2} n_b^{1-B_2} < 0. \tag{5.11}$$

Next, substitute for D_1 and D_2 in the second equation in (5.3), namely $D_1 B_1 n_0^{B_1 - 1} + D_2 B_2 n_0^{B_2 - 1} + \frac{1}{r} = 0$, to get

$$\frac{\lambda^O}{r + \lambda^O} \frac{B_1(1 - B_2)}{B_1 - B_2} \left(\frac{n_0}{n_b}\right)^{B_1 - 1} + \frac{\lambda^O}{r + \lambda^O} \frac{B_2(B_1 - 1)}{B_1 - B_2} \left(\frac{n_0}{n_b}\right)^{B_2 - 1} = 1. \tag{5.12}$$

Equation (5.12) gives us an equation for the ratio $n_0/n_b > 1$. To check that (5.12) has a unique solution greater than 1, note that the left-hand side (1) equals $\lambda^O/(r + \lambda^O) < 1$ when we set $n_0/n_b = 1$, (2) goes to infinity as n_0/n_b goes to infinity, and (3) is strictly increasing with respect to n_0/n_b.

Next, substitute for D_1 and D_2 in the first equation in (5.3), namely $D_1 n_0^{B_1 - 1} + D_2 n_0^{B_2 - 1} + \frac{1}{r} = \frac{1}{n_0}$ to get

$$-\frac{\lambda^O}{r(r+\lambda^O)} \frac{1 - B_2}{B_1 - B_2} \left(\frac{n_0}{n_b}\right)^{B_1 - 1} - \frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1 - 1}{B_1 - B_2} \left(\frac{n_0}{n_b}\right)^{B_2 - 1} + \frac{1}{r} = \frac{1}{n_0}. \tag{5.13}$$

Substitute for n_0/n_b in equation (5.13), and solve for n_0. Finally, we can get n_b from

$$n_b = \frac{n_0}{n_0/n_b}, \tag{5.14}$$

and D_1 and D_2 from equations (5.10) and (5.11), respectively.

Once we have the solution for $\tilde{V}$, we can recover V by

$$\begin{aligned} V(z) &= \max_{n>0} \left[\tilde{V}(n) - zn\right] \\ &= \max_{n>0} \left[D_1 n^{B_1} + D_2 n^{B_2} + \frac{n}{r} - zn\right], \end{aligned} \tag{5.15}$$

in which the critical value n^* solves

$$D_1 B_1 n^{B_1 - 1} + D_2 B_2 n^{B_2 - 1} + \frac{1}{r} = z. \tag{5.16}$$

Thus, for a given value of $z = w/(c-A)$, solve (5.16) for n and plug that value of n into (5.15) to get $\psi(w, A) = V(z)$.

Also of interest is the amount invested in the risky asset, especially as wealth approaches the annuitization level $(c-A)/(r+\lambda^O)$.

$$\pi^*(w, A) = (c-A)\hat{\pi}^*(z) = -(c-A)\frac{\mu - r}{\sigma}\frac{V'(z)}{V''(z)} = -(c-A)\frac{\mu-r}{\sigma}n\tilde{V}''(n).$$

Now, $n\tilde{V}''(n) = D_1 B_1(B_1 - 1)n^{B_1-1} + D_2 B_2(B_2-1)n^{B_2-1}$, so after substituting for D_1 and D_2 from equations (5.10) and (5.11), respectively, the optimal investment in the risky asset (in terms of n) becomes $(c-A)\hat{\pi}^*(z)|_{z=I(-n)} =$

$$(c-A)\frac{\mu - r}{\sigma}\frac{\lambda^O}{r(r+\lambda^O)}\frac{(B_1-1)(1-B_2)}{B_1 - B_2}\left[B_1\left(\frac{n}{n_b}\right)^{B_1-1} - B_2\left(\frac{n}{n_b}\right)^{B_2-1}\right]. \tag{5.17}$$

In particular, as n approaches n_b, the point at which the individual annuitizes all her wealth, the amount invested in the risky asset approaches

$$(c-A)\frac{2r}{\mu - r}\left(\frac{1}{r} - \frac{1}{r+\lambda^O}\right), \tag{5.18}$$

independent of σ and λ^S. Note that the expression in (5.18) is a multiple of the difference between the cost of the perpetuity and the cost of the annuity.

In addition to the amount invested in the risky asset, it is useful to know how that amount changes as one's wealth changes. Note that the derivative of $\pi^*(w, A)$ with respect to w has the same sign as the derivative of $n\tilde{V}''(n)$ with respect to n. Thus, the amount of wealth invested in the risky asset decreases with respect to wealth if and only if

$$n\tilde{V}'''(n) + \tilde{V}''(n) < 0 \text{ for all } n \in (n_b, n_0). \tag{5.19}$$

After some elementary algebra, we determine that (5.19) holds if $\lambda^S < r$, while if λ^S is sufficiently larger than r, then the amount of wealth invested in the risky asset increases with wealth, as we will see in the example in the next section.

5.2. Numerical Examples

In this section, we present numerical examples to demonstrate the results of Section 5.1. We will calculate the probability of lifetime ruin $\psi(w, A)$ in the presence of annuities with the corresponding probability $\psi_0(w, A)$ when the individual cannot buy annuities but has a preexisting income rate of A, the problem studied in Young (2004). From that work, we know that the probability of lifetime ruin $\psi_0(w, A)$ is given by

$$\psi_0(w, A) = (1 - rz)^p, \text{ for } 0 \leq z < \frac{1}{r}, \tag{5.20}$$

in which $z = w/(c - A)$ and

$$p = \frac{1}{2r}\left[(r + \lambda^S + m) + \sqrt{(r + \lambda^S + m)^2 - 4r\lambda^S}\right] > 1. \tag{5.21}$$

Example 5.1, Constant Real Dollar Consumed: Suppose we have the following values of the parameters:

- $\lambda^S = \lambda^O = 0.04$; the force of mortality is constant such that the expected future lifetime is 25 years.
- $r = 0.02$; the riskless rate of return is 2% over inflation.
- $\mu = 0.06$; the drift of the risky asset is 6% over inflation.
- $\sigma = 0.20$; the volatility of the risky asset is 20%.
- $c = 1$; the individual consumes one unit of wealth per year.
- $A = 0$; without loss of generality, we assume that annuity income is zero.

The cost of the annuity is $1/(r + \lambda^O) = 1/0.06 = \$16.666$, while the cost of the perpetuity is flat at $1/r = 1/0.02 = \$50$. In this example, $D_1 = -103.4$, $D_2 = -0.002642$, $n_0 = 0.081$, and $n_b = 0.044$. In Table 1, we give the probabilities of ruin ψ and ψ_0 and the corresponding optimal investments in the risky asset as a proportion of $(c - A) = 1$, i.e., $\hat{\pi}^*$ and $\hat{\pi}_0^*$.

Table 1. Probability of Lifetime Ruin and Optimal Investment in Risky Asset

z	$\psi(w, A) = V(z)$	$\hat{\pi}^*(z)$	$\psi_0(w, A) = V_0(z)$	$\hat{\pi}_0^*(z)$
0.0	1.000	25.283	1.000	20.711
0.5	0.960	25.300	0.966	20.504
1.0	0.921	25.327	0.933	20.296
2.0	0.844	25.415	0.870	19.882
5.0	0.633	25.977	0.698	18.640
7.5	0.474	26.829	0.574	17.604
10.0	0.330	28.066	0.467	16.569
12.0	0.223	29.345	0.392	15.740
14.0	0.123	30.885	0.326	14.912
16.0	0.030	32.680	0.268	14.083
16.5	0.0074	33.168	0.255	13.876
16.6	0.00296	33.267	0.252	13.835
16.66	0.000296	33.327	0.251	13.810
16.666	0.0000296	33.333	0.251	13.807
20.0	0.000	n.a.	0.175	12.426

There are a variety of interesting lessons that can be gleaned from the numbers in Table 1. First, for very low values of z, the ratio of current wealth to the desired additional consumption, the probability of lifetime ruin is (obviously) close to 100%, but it is quite insensitive to whether or not annuities are available. Intuitively, the reason is that the costs of the annuity and the perpetuity are both relatively far from current wealth and are therefore probabilistically inaccessible. However, as the value of z increases, the probability of lifetime ruin starts to decline, and the rate of probability improvement is much higher when the life annuity is available. In fact, as we get very close to the cost of the annuity, \$16.666, the probability of lifetime ruin approaches zero – since as soon as that level is reached the entire wealth will be annuitized – while the perpetuity cost is still a distance away at \$50.

As predicted by equation (5.18), as we get (epsilon) close to the annuity cost – i.e. when z = 16.5, 16.6 etc. – we see the equity allocation move towards 33.333 = 50.00 - 16.666, the difference between the cost of the perpetuity and the cost of an annuity. Another use of the results in Table 1 is to invert the ψ function and solve for the current level of wealth-to-consumption needed to maintain a lifetime ruin probability under some pre-specified level. Thus, for example, if the retiree is interested in having at least a 95% chance of lifetime consumption survival – which implies at most a 5% probability of lifetime ruin – then she must have wealth of at least z = 15.55 times her desired consumption.

Note from Table 1, that $\psi(w, A) < \psi_0(w, A)$, as expected, because the probability of lifetime ruin should decrease as the individual's investment opportunities expand to include annuities. On the other hand, the optimal investment in the risky asset increases with respect to z in the presence of annuities for this particular example (that is, $\lambda^S = 0.04$ is sufficiently larger than $r = 0.02$), but it decreases when the individual cannot buy annuities. Young (2004) showed the latter, but $\hat{\pi}^*$ might increase or decrease when the individual can buy annuities, depending on the magnitude of λ^S relative to r; see the discussion immediately following (5.19). Note also that we have $\hat{\pi}^*(z) > z$; thus, the individual borrows in order to invest in the risky asset. In particular, at the lower wealth levels, the optimal strategy is a heavily leveraged position in the risky asset. This is because of the binary nature of the investor's goal and because $\hat{\pi}$ is unconstrained in our problem formulation. We comment on the constrained problem in Section 7.

The following table illustrates the impact of life expectancy on the ruin probability and the optimal allocation to equity. We present three different cases. The first is for a life expectancy of 15 years ($\lambda = 0.066$), the second is a life expectancy of 20 years ($\lambda = 0.05$), and the final is the above-calculated case ($\lambda = 0.04$). In each case, $\lambda^S = \lambda^O = \lambda$.

Table 2. The Impact of Life Expectancy on Lifetime Ruin and Optimal Investment

	$1/\lambda = 15$		$1/\lambda = 20$		$1/\lambda = 25$	
Wealth to Consump'n Ratio (z)	Ruin Probability $\psi(w, A)$	Equity Investment $\hat{\pi}^*(z)$	Ruin Probability $\psi(w, A)$	Equity Investment $\hat{\pi}^*(z)$	Ruin Probability $\psi(w, A)$	Equity Investment $\hat{\pi}^*(z)$
10.0	0.111	34.980	0.248	29.978	0.330	28.066
12.0	0.000	n.a.	0.128	32.476	0.223	29.345
14.0	0.000	n.a.	0.016	35.289	0.123	30.885
16.0	0.000	n.a.	0.000	n.a.	0.030	32.680
18.0	0.000	n.a.	0.000	n.a.	0.000	n.a.

Notice from Table 2 that as the individual's life expectancy increases, the amount invested in the risky asset decreases and the probability of lifetime ruin increases, for a given level of z.

Example 5.2, Benefit of Dynamic Portfolio Management and Annuity Purchase: Recall that in Section 2.1, we computed the probability of lifetime ruin under the assumption of exponential future lifetime for an individual with wealth \$1 who invests in the risk-free asset only and self-annuitizes; i.e., who consumes $c = \lambda + r$ per year, the amount of life-annuity income that \$1 would provide. In this example, we quantify the benefit of dynamic portfolio management and the purchase of a life-annuity by computing the probability of lifetime ruin for an individual who consumes $c = \lambda + r$ per year and by contrasting the results with those of Section 2.1.

As in Example 5.1, we choose $\lambda^S = \lambda^O = \lambda = 0.04, r = 0.02, \mu = 0.06$, and $\sigma = 0.2$. In the deterministic case, an individual with initial wealth \$1 who self-annuitizes consumes $c = \lambda + r = 0.06$ per year. By the results of Section 2.1, we have that the ruin probability is $\left(1 + \frac{r}{\lambda}\right)^{-\frac{\lambda}{r}} = 0.4444$.

If life annuities are available, an investor with wealth \$1 can purchase a life annuity to provide the desired income $c = \lambda + r$, therefore the ruin probability ψ is zero when wealth equals \$1. Moreover, Figure 5.1 shows that $\psi(w) < 0.444$ for $w \in (0.5, 1)$. Thus, dynamic portfolio management and a life annuity purchase yield lower ruin probability, even with lower wealth.

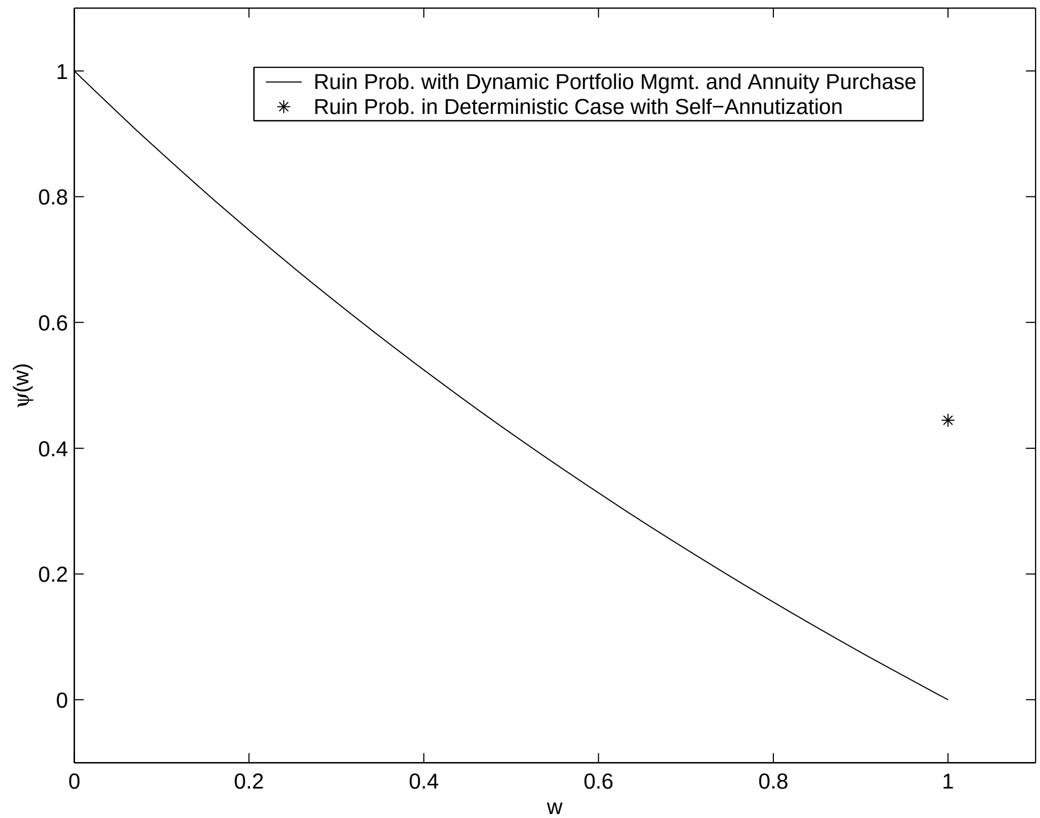

Figure 5.1: With dynamic portfolio management and a life-annuity purchase, an individual can maintain the same consumption level with lower ruin probability, even with lower wealth.

6. Solution of the Free-Boundary Problem: General Force of Mortality

In the previous section, we observed that if we assume constant force of mortality, we can derive an implicit analytical solution to the free-boundary problem. Under more general mortality assumptions, there is no implicit analytical solution to the free-boundary problem given by (4.8), (4.11), and (4.12).

In this section, we exploit the connection between solutions of free-boundary problems and value functions for optimal stopping problems (Øksendal, 1998). We recast our free-boundary problem as a variational inequality for the value function of an optimal stopping problem.

We employ the projected SOR method (Wilmott, Dewynne, and Howison, 2000) to calculate the solution of the free-boundary problem numerically. We demonstrate that in the case of constant force of mortality, the results of our numerical method agree with those in Example 5.1. We, then, consider several examples in which we examine the effect on the ruin probability and on the optimal investment strategy of changing the mortality assumptions and the parameters of the financial model.

6.1. Optimal Stopping Formulation

In this section, we propose an optimal stopping problem whose value function $\hat{f}$ corresponds with the solution $\tilde{f}$ of the free-boundary problem (4.8), (4.11), and (4.12). Equations (4.11) and (4.12) motivate us to define a penalty function u by

$$u(y,t) = \min({}_tp_x^S, \bar{a}_{x+t}^O y). \tag{6.1}$$

We consider this function because it is maximal among those functions that are concave in y and satisfy the boundary conditions in (4.11) and (4.12). Recall that $\tilde{f}$ is concave and increasing in y. Thus, $\tilde{f}(y,t) \le u(y,t)$ for all (y,t) such that $y_b(t) \le y \le y_0(t)$.

Define a stochastic process Y_s by

$$\begin{cases} dY_s = -rY_s + \dfrac{\mu - r}{\sigma} Y_s d\tilde{B}_s \\ Y_t = y > 0. \end{cases} \tag{6.2}$$

Finally, consider the optimal stopping problem given by

$$\hat{f}(y,t) = \inf_{\tau} E\left[\int_t^{\tau} Y_s ds + u(Y_\tau, \tau) | Y_t = y\right]. \tag{6.3}$$

One can think of this problem as awarding a "player" the running penalty Y_s between time t and the time of stopping τ. At the time of stopping, the player receives the penalty $u(Y_\tau, \tau)$. Thus, at each point in time, the player has to decide whether it is better to continue receiving the running penalty Y_s or to stop and take the final penalty $u(Y_\tau, \tau)$.

By Øksendal (1998, Chapter 10), the value function $\hat{f}$ of the optimal stopping problem solves the variational inequality

$$\max\left[-\hat{f}_t + ry\hat{f}_y - my^2\hat{f}_{yy} - y, \hat{f} - u\right] = 0, \tag{6.4}$$

A candidate solution for $\hat{f}$ is the value function $\tilde{f}$ from Section 4. Indeed, Øksendal (1998, Section 10.4) studies such optimal stopping problems and proves a verification theorem that we can apply as follows: If we can show that

$$u_t(y,t) - ryu_y(y,t) + my^2u_{yy}(y,t) + y \geq 0, \tag{6.5}$$

for $y > y_0(t)$ and for $y < y_b(t)$, and if $\tilde{f}$ is sufficiently regular (smooth, etc.), then $\hat{f} = \tilde{f}$. Thus, to numerically solve for $\tilde{f}$, we can use algorithms developed for optimal stopping problems and solve for $\hat{f}$, the value function of the optimal stopping problem.

It remains for us to verify that inequality (6.5) holds. Indeed, for $y < y_b(t)$, we have that $u(y,t) = \bar{a}^O_{x+t}y$, so that

$$\begin{aligned} &u_t(y,t) - ryu_y(y,t) + my^2u_{yy}(y,t) + y \\ &\quad = [-1 + (r + \lambda^O_{x+t})\bar{a}^O_{x+t}]y - ry\bar{a}^O_{x+t} + y \\ &\quad = \lambda^O_{x+t}\bar{a}^O_{x+t}y \geq 0, \end{aligned} \tag{6.6}$$

so (6.5) holds here. For $y > y_0(t)$, we have that $u(y,t) = {}_tp^S_x$, so that

$$\begin{aligned} &u_t(y,t) - ryu_y(y,t) + my^2u_{yy}(y,t) + y \\ &\quad = -\lambda^S_{x+t}\ {}_tp^S_x + y, \end{aligned} \tag{6.7}$$

and this expression is nonnegative for all $y > y_0(t)$ if and only if

$$y_0(t) \geq \lambda^S_{x+t}\ {}_tp^S_x. \tag{6.8}$$

Inequality (6.8) holds if the pricing force of mortality is increasing, and if $\lambda^S_{x+t} \leq r + \lambda^O_{x+t}$. Indeed, $y_0(t) = -f_z(0,t)$. By convexity of f, the slope of f at $(0, {}_tp^S_x)$ is less than the slope of the secant from $(0, {}_tp^S_x)$ to $(\bar{a}^O_{x+t}, 0)$. That is, $f_z(0,t) \leq -{}_tp^S_x/\bar{a}^O_{x+t}$, which implies that $y_0(t) \geq {}_tp^S_x/\bar{a}^O_{x+t} \geq \lambda^S_{x+t}\ {}_tp^S_x$ because λ^O_{x+t} is increasing.

6.2. The Numerical Method

In this section, we briefly describe the numerical treatment of the variational inequality (6.4). Because (6.4) is similar to the variational inequality associated with pricing an

American option, we employ the projected SOR method to find the solution $\hat{f}$ of (6.4) and to recover the free boundary. Øksendal (1998, Section 10.4) ensures that $\hat{f}$ also solves the free-boundary problem (4.8), (4.11), and (4.12).

To employ the Projected SOR method, we

1. Transform the degenerate problem in (6.4) on $(0, \infty)$ to a non-degenerate problem on $\mathcal{R}$ via the standard transformation $\xi = \ln y$.

2. Solve the transformed variational inequality via the projected SOR method and recover the location of the free boundary.

3. Invert the dual transform as in Section 5 to recover $V(z,t)$ from $\hat{f}(z,t)$.

4. Numerically approximate the optimal investment in the risky asset $\pi^*(z,t)$.

Figure 6.0 shows the results of the numerical method described in this section for Example 5.1 and compares them to the analytical solution that was given there. The first graph shows the increasing, concave solution to the dual variational inequality (6.4). The free boundary separates the regions $\{\hat{f} = u\}$ and $\{\hat{f} < u\}$, in which u is the piecewise linear function with slopes $\bar{a}^O_{x+t}$ and 0 defined in (6.1). Stated another way, $y_b(t) = \inf\{y : \hat{f}_y(y,t) < \bar{a}^O_{x+t}\}$, and $y_0(t) = \inf\{y : \hat{f}_y(y,t) = 0\}$.

The second graph in Figure 6.0 shows the decreasing, convex ruin probability $V(z, 0)$. We note that, consistent with the model formulation, $V(0,t) = 1$ and $V(\bar{a}^O_{x+t}, t) = V(16.67, t) = 0$ for $t = 0$. The third graph shows the optimal investment in the risky asset. We note that our computed solutions match those given in Example 5.1.

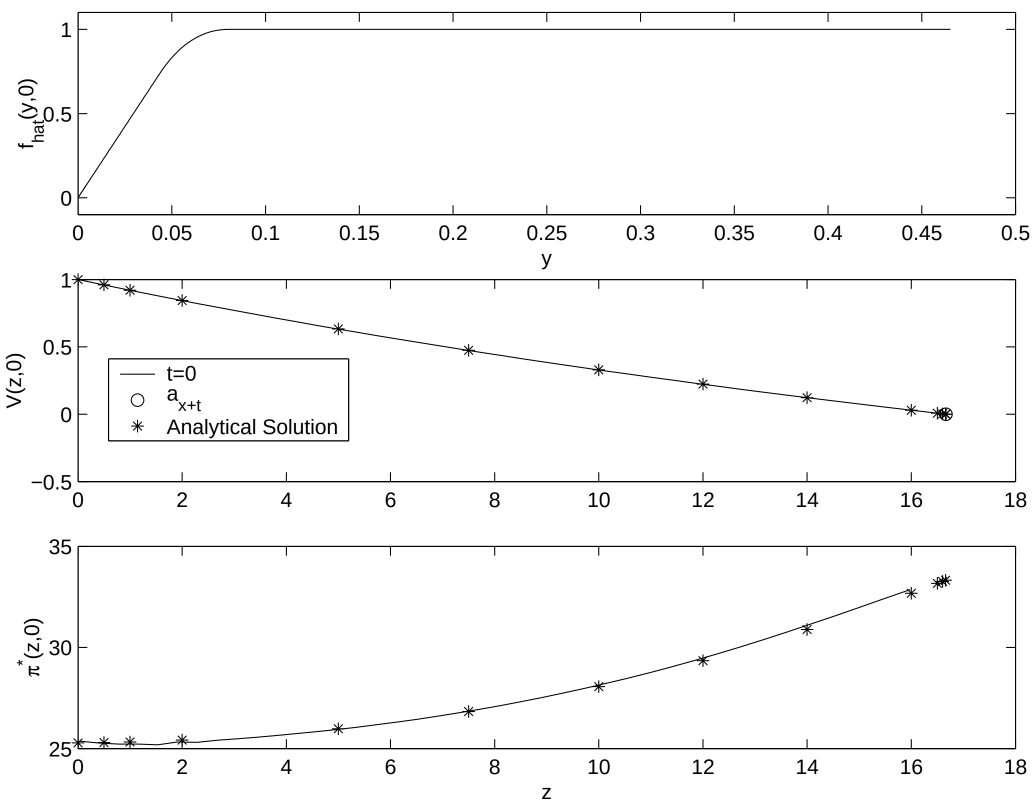

Figure 6.0: Our numerical results are consistent with the model formulation and the analytical solution given in Example 5.1

6.3. Examples

In this section, we consider several examples. We begin with a base scenario and then examine the effect on the ruin probability and optimal investment strategy of changing the mortality assumptions and the parameters of the financial model. We take the following as our base scenario:

Base Scenario:

• Consistent with the mortality assumptions in Milevsky and Young (2003) and Huang, Milevsky, and Wang (2004), we use the Gompertz hazard rate $\lambda_x^O = \exp\left(\frac{(x-\bar{m})}{b}\right)/b$, where $\bar{m}$ is a modal value and b is a scale parameter. Note that the force of mortality increases exponentially with age. We choose $\bar{m} = 90$ and $b = 9$. Also, let $\lambda_x^S = \lambda_x^O + \eta$, where η is a parameter that quantifies the individual's mortality relative to the pricing mortality. To begin, we let $\eta = 0$.

• $x = 50$; the investor is 50 years old.

• $r = 0.02$; the riskless rate of return is 2% over inflation.

• $\mu = 0.06$; the drift on the risky asset is 6% over inflation.

- $\sigma = 0.20$; the volatility of the risky asset is 20%.

- $c = 1$; the individual consumes one unit of wealth per year.

- $A = 0$; without loss of generality, we assume that annuity income is zero.

In the experiments that follow, we will examine the impact on the ruin probability and optimal investment strategy of varying individual parameters from the values given above.

Example 6.1, Impact of Attained Age: Figure 6.1 shows the ruin probability $V(z, 0)$ and optimal investment in the risky asset $\pi^*(z, 0)$ for the base scenario described above as well as for ages 30 and 70. We note that, consistent with the model formulation, $V(0, 0) = 1$ and $V(\bar{a}_x^O, 0) = 0$. Moreover, we see that the ruin probability and optimal investment in the risky asset decrease as age increases. Thus, a younger investor with wealth $z_0 \in (0, \bar{a}_x^O)$ is more likely to ruin than an older investor with the same wealth. In addition, the younger individual will invest more in the risky asset than an older individual with the same wealth. This result is consistent with our financial intuition. Note that for some z, because of the all-or-nothing nature of the investor's objective and because we did not constrain π^* in our problem formulation, the investment in the risky asset exceeds current wealth.

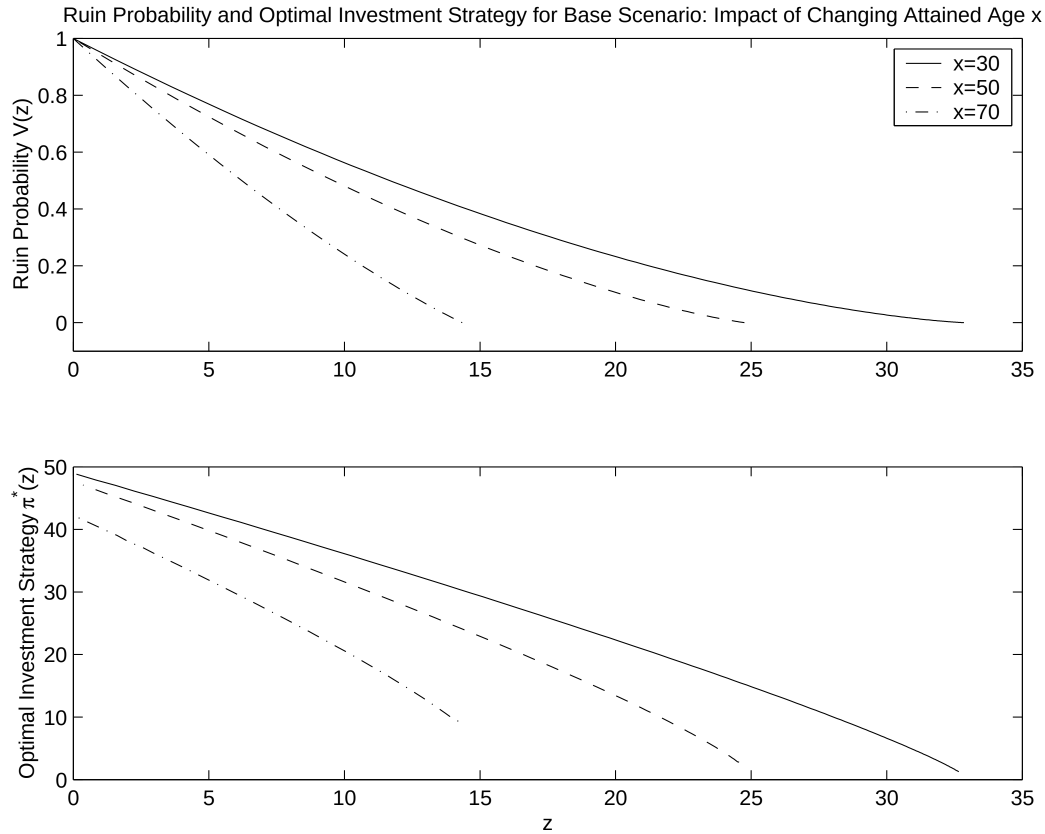

Figure 6.1: Ruin probabilities and optimal investment strategies as we vary the attained age x

Example 6.2, Impact of Stock Volatility: In Figure 6.2, we examine the impact of changing the volatility σ of the stock return. We observe that for fixed z, the ruin probability increases and the optimal investment in the risky asset decreases with σ. Again, this is consistent with our financial intuition.

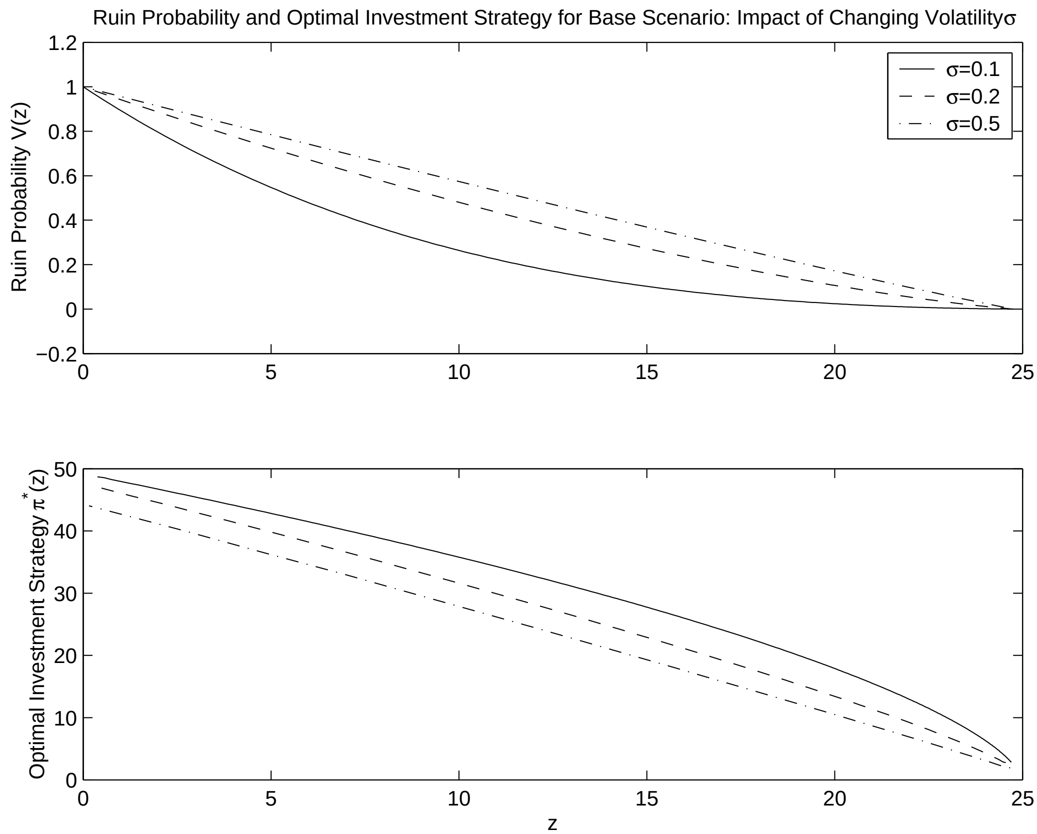

Figure 6.2: Ruin probabilities and optimal investment strategies as we vary the volatility σ

Example 6.3, Impact of Individual-Specific Mortality (Gompertz): In Figure 6.3, we examine the impact of individual-specific mortality on the ruin probability and optimal investment strategy. We use the Gompertz assumption described above as the pricing mortality λ_x^O and we define the individual-specific mortality by $\lambda_x^S = \lambda_x^O + \eta$ for $\eta = -0.005, \eta = 0$, and $\eta = 0.005$. (For this example, we consider a 65-year-old investor in order to avoid negative force of mortality.) The first graph in Figure 6.3 shows the individual-specific forces of mortality and the second graph shows the corresponding survival probabilities. In the third graph, we see that individual-specific mortality has little impact on the probability of lifetime ruin. This occurs because, as the fourth graph shows, the investor adjusts her investment strategy compensate for the change in mortality; in effect, the change in strategy neutralizes the impact on the ruin probability of the change in subjective mortality. Note that an individual with lower mortality (and thus a longer investment horizon) invests more in the risky asset.

Huang, Milevsky, and Wang (2004) examined lifetime ruin probability for an individual who invests in the risky asset only and who purchases no life annuities. They show that the probability of lifetime ruin for a 65-year-old with initial wealth \$20 who consumes $c = 1$ per year is approximately 0.57. They also show that in order to sustain annual consumption of $c = 1$ with a ruin probability of 0.05, a 70-year-old requires initial wealth of \$27. In our model, a 65-year-old with only \$17.05 can purchase a life annuity to provide the desired consumption $c = 1$ with zero probability of ruin. Moreover, a 65-year-old individual needs only \$15.67 to sustain consumption of $c = 1$ with ruin probability 0.05. Thus we see that, with dynamic portfolio management and life annuities, one can sustain the desired level of consumption with the same (or lower) ruin probability with lower wealth. (We remark that the parameter values in Huang, Milevsky, and Wang (2004) differ slightly from ours, but this does not change the qualitative comparison of the results.)

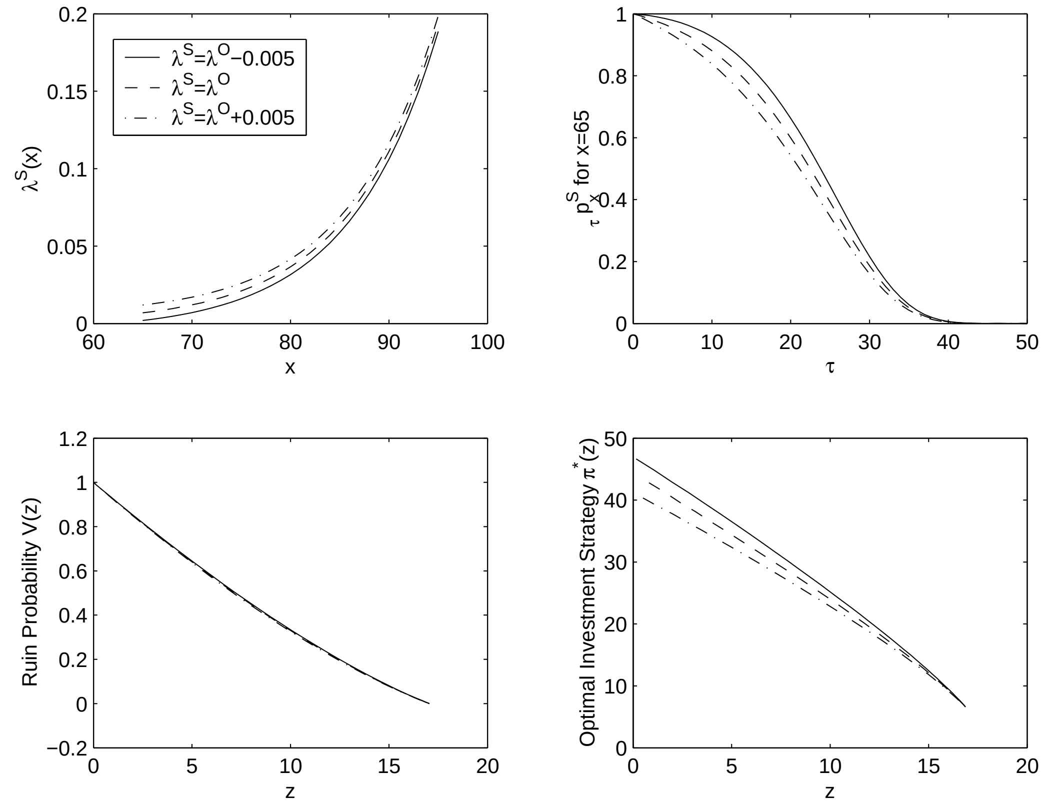

Figure 6.3: Gompertz mortality for 65-year-old: Impact of individual-specific mortality

Example 6.4, Impact of Individual-Specific Mortality (Constant Force): We repeat the analysis of Example 6.3, but with constant force of mortality instead of the Gompertz hazard rate. We use pricing mortality $\lambda^O = 0.04$ and consider an individual whose specific mortality is given by $\lambda^S = \lambda^O + \eta$ for $\eta = -0.015, \eta = 0$, and $\eta = 0.015$. We see in Figure 6.4 that the effect on the investment strategy is more pronounced than under Gompertz mortality. In particular, for $\lambda^S = 0.055$ and $\lambda^S = 0.04$, the optimal investment in the risky asset increases with wealth. For $\lambda^S = 0.025$, it decreases with wealth. In Example 6.3, under Gompertz mortality, the optimal investment in the risky asset decreased with wealth regardless of λ^S.

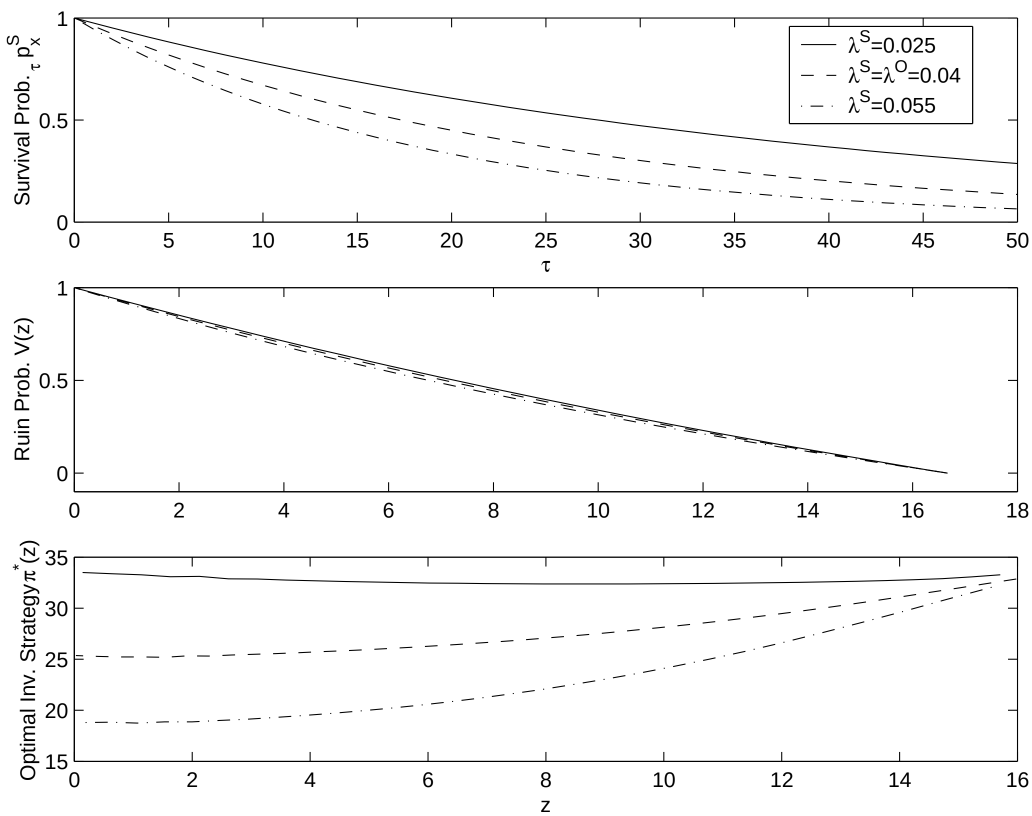

Figure 6.4: Constant force of mortality: Impact of individual-specific mortality

Example 6.5, Impact of Pricing Mortality: In Figure 6.5, we examine the impact of changing the objective (pricing) mortality assumption for a 50-year-old whose specific mortality λ^S equals the pricing mortality λ^O. More specifically, we examine the ruin probability and optimal strategy under two different mortality assumptions that yield the same price for a life annuity. For our base scenario, we assume Gompertz mortality with the parameters given above. Under this assumption, the price of the annuity is $\bar{a}_{50}^O = 24.75$. We contrast the ruin probability and optimal strategy under Gompertz mortality with the results under constant force of mortality with $\lambda^S = \lambda^O = 0.0204$ (so that $\bar{a}_{50}^O = 24.75$). The first two graphs in Figure 6.5 show the forces of mortality and corresponding survival probabilities. The fourth graph shows that the change in the mortality assumption has a dramatic impact on the optimal investment strategy. Thus, although both assumptions yield the same annuity price, the shape of the hazard rate has a significant effect on the optimal strategy. Under Gompertz mortality, the individual invests more in the risky asset at lower wealth levels because of the higher survival probability in the early years. As in the previous experiment, there is little change in the probability of ruin.

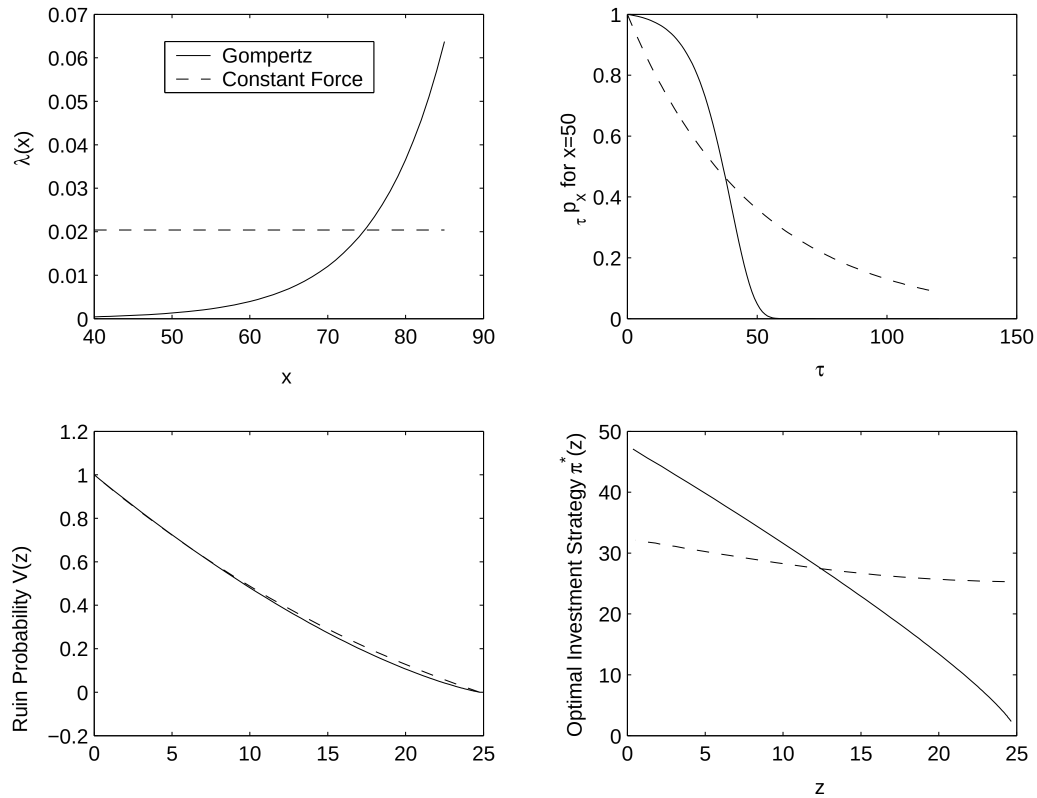

Figure 6.5: Both mortality assumptions above yield the same annuity price, but the shape of the hazard rate has significant impact on the optimal investment strategy.

Example 6.6, Impact of the Availability of Life Annuities: Moore and Young (2004) study minimal ruin probability and optimal investment strategies in a market in which life annuities are not available. In Figure 6.6, we compare the ruin probability and optimal investment strategy with those quantities for an investor for whom life annuities are not available. We see that the presence of annuities decreases the probability of lifetime ruin. Indeed, when $z = \bar{a}_x^O \approx 24.75, V(z, 0) = 0$; in a market with no annuities, an investor requires greater wealth to ensure that $V(z, 0) = 0$. This is consistent with the results in Table 1 of Example 5.1.

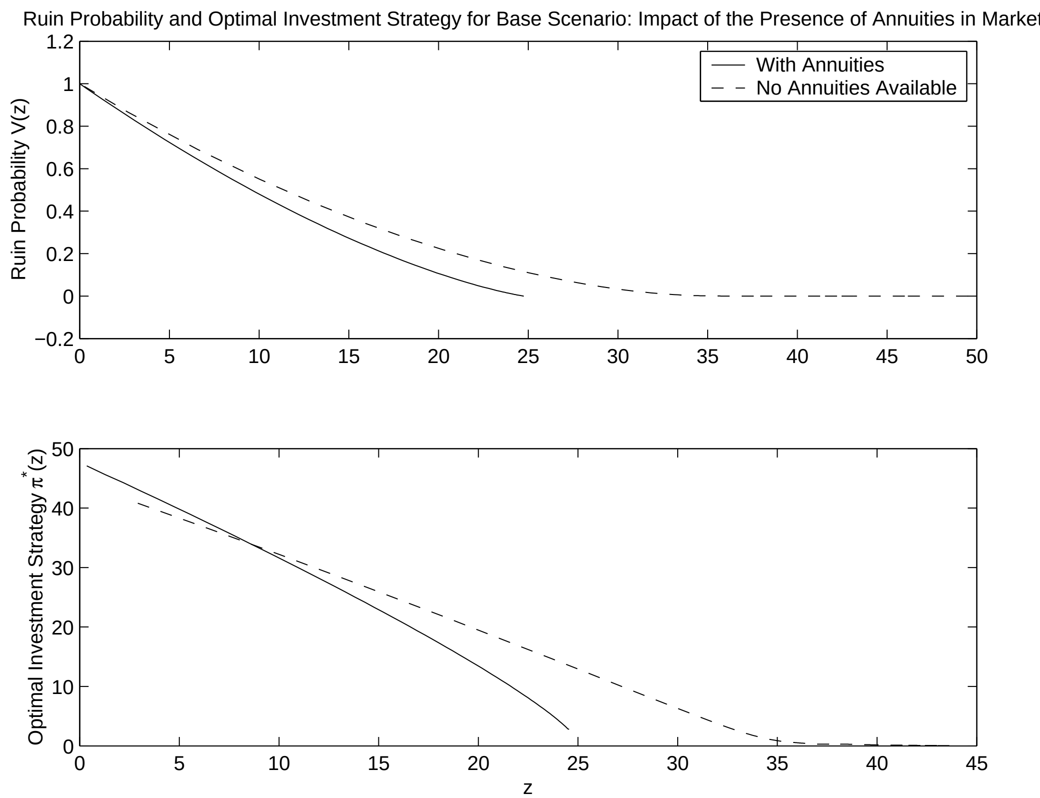

Figure 6.6: The impact on the ruin probability and optimal investment strategy of the inclusion of annuities in the market

7. Conclusion and Future Research

In this paper, we derived the optimal investment and annuitization strategy for a retiree whose objective is minimize the *probability of lifetime ruin*, the probability that a fixed consumption strategy will lead to zero wealth while the individual is still alive. We obtained a variety of interesting results. First, given the all-or-nothing objective, we found that the ruin-minimizing annuitization strategies are of the bang-bang, as opposed to gradual, type. In several of the numerical examples, we saw that the ruin probability and optimal strategies respond in an intuitive and predictable way to changes in the model parameters. However, the impact of the mortality assumption, and in particular, the shape of the hazard rate function, is significant. Under some mortality assumptions, the optimal investment in the risky asset increases with wealth, while under other assumptions, it decreases.

Finally, we note that we considered an unconstrained optimization problem; we did not restrict the investment in the risky asset to be less than or equal to current wealth. Because of this and because of the binary nature of the investor's objective, in several examples, the optimal strategy was a heavily leveraged position in the risky asset. We plan to address this by considering the constrained problem in future work.

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