

Annuity and Asset Allocation

Moshe A. Milevsky¹ and Virginia R. Young²
York University and University of Michigan

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¹Milevsky, the contact author, is an Associate Professor of Finance at the Schulich School of Business, York University, Toronto, Ontario, M3J 1P3, Canada, and the Director of the Individual Finance and Insurance Decisions (IFID) Centre at the Fields Institute. He can be reached at Tel: (416) 736-2100 ext 66014, Fax: (416) 763-5487, E-mail: milevsky@yorku.ca. This research is partially supported by a grant from the Social Sciences and Humanities Research Council of Canada and the Society of Actuaries.

²Young is a Professor of Mathematics at the University of Michigan, Ann Arbor, Michigan, 48109, USA. She can be reached at Tel: (734) 764-7227, Fax: (734) 764-7048, E-mail: vryoung@umich.edu. The authors acknowledge helpful comments from Ravi Jagannathan (the editor), the anonymous *Management Science* reviewer as well as Jeff Brown, Narat Charupat, Glenn Daily, Jerry Green, Mike Orszag, Mark Warshawsky, and seminar participants at *York University, University of Michigan, University of Wisconsin, University of Cyprus, the American Economics Association* and the *Western Finance Association* annual meeting. Please note that the current version of this paper is a combination of two previous working papers that were independently circulated under the titles: *The Real Option to Defer Annuity: It's Not Now-or-Never* and *Optimal Annuity Purchasing*.

Abstract

This paper examines the optimal annuitization as well as the usual investment and consumption strategies of a utility-maximizing *retiree* facing a stochastic time of death under a variety of institutional pension and annuity arrangements. We focus on the impact of aging and the increase in the actuarial force of mortality on the optimal purchase of mortality-contingent annuities, which form the basis of most Defined Benefit (DB) pension plans. Due to severe adverse selection concerns, acquiring a lifetime payout annuity is an irreversible transaction that, we argue, creates an incentive to delay. Under the institutional all-or-nothing arrangement where annuitization must take place at one distinct point in time (i.e. retirement) we derive the optimal age at which to annuitize and present a metric to capture the loss from annuitizing prematurely. In contrast, under an open-market structure where individuals can annuitize a fraction of their wealth at distinct points in time, we locate a general optimal annuity purchasing policy. In this case, we find that an individual will initially annuitize a lump-sum and then buy annuities in order to keep wealth to one side of a separating *ray* in wealth-annuity space, a type of barrier control result. We believe our paper is the first to present an integrated optimal policy for annuitization in the presence of realistic institutional restrictions.

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1 Introduction and Motivation

Asset allocation and consumption decisions towards the end of the human life cycle are complicated by the uncertainty associated with the length of life. Although this risk can be completely hedged in a *perfect market* with life annuities – or more precisely with continuously renegotiated tontines – real world frictions and imperfections impede the ability to do so in practice. Indeed, empirical evidence suggests that voluntary annuitization amongst the public is not very common, nor is it well understood even amongst the financial experts. Therefore, in attempt to fill this gap and integrate mortality-contingent claims into the finance literature, this paper examines the optimal annuitization strategy of a utility-maximizing *retiree* facing a stochastic time of death under a variety of institutional pension and annuity arrangements. We also examine the usual investment and consumption dynamics but focus our attention on the impact of aging and the increase in the actuarial force of mortality on the optimal purchase of mortality-contingent annuities, which form the basis of most Defined Benefit (DB) pension plans. Our main insight is that due to severe adverse selection concerns, acquiring a lifetime payout annuity is an irreversible transaction that, we argue, creates an incentive to delay. In fact, we employ the pedagogical analogy of a classical American option which should only be exercised once the value from waiting is zero.

For the most part of the paper, the focus of our attention is a life annuity that pays a fixed (real or nominal) continuous payout for the duration of the annuitant's life. From a financial perspective, this product is akin to a coupon-bearing bond that defaults upon death of the holder and for which there is no secondary market. Under the institutional all-or-nothing arrangement where annuitization (i.e. the purchase) must take place at one distinct point in time (i.e. retirement) we locate the optimal age at which to annuitize and present a metric to capture the loss from annuitizing prematurely. This optimal age, which is linked to the actuarial force of mortality, occurs well within the classical retirement years and is obviously gender specific but also depends on the individual's subjective health status. All of this will be explained in the body of the paper.

In contrast to the restrictive (yet quite common) all-or-nothing arrangement, under an open-market structure where individuals can annuitize a fraction of their wealth at distinct points in time, we locate a general optimal annuity purchasing policy. In this case, we find that an individual will initially annuitize a lump sum – if they do not already have this minimum level in pre existing DB pensions – and then buy additional life annuities in order to keep wealth to one side of a separating *ray* in wealth-annuity space. This is a type of barrier control result which is common in the literature on asset allocation with transaction costs.

In sum, we believe our paper is the first to present an integrated asset allocation and consumption policy in continuous time taking into account life annuity products in the presence of realistic institutional restrictions.

1.1 Agenda and Outline

The remainder of this paper is organized as follows. In Section 2 we provide a brief explanation of the mechanics of the life annuity market and review the existing literature involving asset allocation, personal pensions, and payout annuities. In Section 3 we present the general model for our financial and annuity markets. In Section 4 we consider the case for which the individual is required to annuitize *all* her pensionable wealth at one point in time. This is effectively an optimal retirement problem and is akin to the situation in the United Kingdom where retirees can drawdown their pension but must annuitize the remaining balance by a certain age or where individuals have the choice of when to start their retirement (DB) pension but must do so at *one* point in time. In fact, most Variable Annuity contracts sold in the United States have an embedded option to annuitize that can only be exercised once. In this restrictive (but common) framework we locate the optimal age for her to do so, and then define a so-called option value *metric* as the gain in utility from annuitizing optimally. Section 5 provides a variety of numerical examples for the optimal time to annuitize and also pursues the option analogy as a way of illustrating the loss from annuitizing pre-maturely. Then, in Section 6 we consider a less restrictive open-market arrangement whereby the individual may annuitize any portion of her wealth at any point time. This would be applicable to individuals with substantial discretionary wealth who can purchase small (or large) quantities of annuities on an ongoing basis. In this case, we find that the individual annuitizes a minimal amount at retirement and then slowly acquires more annuities depending on the performance of their stochastic wealth process. If her wealth subsequently increases in value, she purchases more annuities by annuitizing additional wealth, otherwise she refrains from additional purchases and consumes from her originally-purchased annuities as well as from liquidating investments in her portfolio. Furthermore, we *explicitly* solve for the optimal annuity purchasing policy under this less restrictive case when the force of mortality is constant, which implies that the future lifetime is exponentially distributed. Non-essential proofs and theorems are relegated to an Appendix (Section 8), while Section 7 concludes the paper with our main qualitative insights.

2 The Annuity Market and Literature

2.1 Life Annuity Basics

Life annuities are purchased directly from insurance companies but actually form the basis of most Defined Benefit pension plans. In exchange for a lump-sum premium which the company invests in it's general account the company guarantees to pay the annuitant a fixed (monthly or quarterly) payout for the rest of his or her life. This payout rate – which depends on prevailing interest rates and mortality projections – is irrevocably determined at the time of purchase (a.k.a. annuitization) and does not change for the life of the contract. The following chart illustrates some sample quotes which were provided by a life-annuity broker (in Canada).

Certain \ Age	m 55 f		m 60 f		m 65 f		m 70 f		m 75 f		m 80 f	
0 yrs	631	590	686	633	765	694	877	780	1039	911	1259	1096
5 yrs	628	589	681	631	755	689	855	770	989	887	1146	1036
10 yrs	620	584	666	623	726	674	799	741	879	825	940	901
15 yrs	607	578	644	611	687	652	729	699	764	745	774	765
20 yrs	591	569	618	596	643	625	662	651	673	668	665	664
25 yrs	573	559	589	578	601	594	608	605	610	609	N.A.	N.A.
Sample monthly payout based on a \$100,000 initial premium (purchase)												

For example, in exchange for a \$100,000 initial premium, a 75 year-old female would be guaranteed to receive \$911 per month for the rest of her life. This life annuity would have no guarantee period, which means that if she were to die one instant after purchasing the life annuity (technically it would have to be after the first payment) her beneficiaries or estate would receive nothing in return. The \$911 monthly income for those who survive, consists of a blended mix of principal and interest as well as the implicit funds of those who do not survive. Note that a male would receive slightly more per month, namely \$1,039, due to their lower life expectancy.

A few things should be obvious from the table. First, the higher the purchase age, all else being equal, the greater the annuity income. Naturally, the life expectancy is shorter and the initial premium must be amortized and returned over a shorter time period. Likewise, a longer guarantee period yields a lower annuity income. In fact, an 80 year-old requesting a 20 year guarantee will receive virtually the same amount (\$664) as a male of the same age, since neither is likely to live past the 20 year certain period, and hence the annuity is essentially a portfolio of zero coupon bonds. Some other points are in order:

1. The law of large numbers and the ability to diversify away all mortality risk is central to the pricing of life annuities. The above-mentioned payouts are determined by expected *objective* annuitant mortality patterns together with prevailing (risk-free) interest rates of the same duration. Profits, fees and commissions are built into these quotes by *loading* the pure actuarial factor on the order of 1% to 5%.
2. Payout rates fluctuate from week-to-week because most of the insurance companies assets backing these lifetime guarantees are invested in fixed-income instruments. This can sometimes cause quotes to change on a daily basis. It is therefore not a stretch to model the evolution of these prices in continuous time.
3. Most of the existing open-purchase annuity market in North America is based on fixed nominal (and not inflation adjusted) payouts. Indeed, real annuities are quite rare, which is an ongoing puzzle to many economists. Consequently, the numerical examples in our paper will focus on nominal values, although there is obviously nothing in our model that precludes using inflation-adjusted prices and returns.
4. An additional form of life annuity is the variable payout kind whose periodic income is linked to the performance of pre-selected equity and bond indices. In this case, the above-mentioned \$100,000 premium would go towards purchasing a number of payout *units* (as opposed to dollars) whose value would fluctuate over time. These annuities are the foundation of the US-based TIAA-CREFs pension plan for University workers, but are quite rare anywhere else in the world. As a result, the bulk of our paper addresses the fixed payout kind, but we refer the interested reader to appendix (8.4) in which these products are integrated into our model.

2.2 Literature Review

This paper merges a variety of distinct strands in the literature. First, our work sits squarely within the classical Merton (1971) optimal asset allocation and consumption framework. However, in contrast to extensions of this model by Kim and Omberg (1996), Koo (1998), Sorensen (1999), Wachter (2002) or the recent book by Campbell and Viceira (2002), for example – which are all concerned with relaxing the dynamics of the underlying economic state variables and investigating the impact of time horizon on portfolio choice – our model attempts to realistically incorporate mortality-contingent payout annuities within this framework.

A life-contingent annuity is the building block of most defined benefit (DB) pension plans – see Bodie, Marcus and Merton (1988) for details – but can also be purchased in the open market. The irreversibility of this purchase is due to the well-known adverse selection issues identified by Akerlof (1970) and Rothschild and Stiglitz (1976).

On its own, the topic of payout annuities has been investigated quite extensively within the public economics literature. In fact, a so-called annuity puzzle has been identified in this field. The puzzle relates to the incredibly low levels of voluntary annuitization exhibited by retirees who are given the choice of purchasing a mortality-contingent payout annuity. For example, holders of *variable annuity saving* policies in the U.S. have the option to convert their accumulated savings into a payout annuity, and yet less than 2% elect to do so according to the National Association of Variable Annuities. In the comprehensive Health and Retirement Survey (HRS) conducted in the U.S, only 1.57% of the HRS respondents reported annuity income. Likewise, only 8.0% of respondents with a defined contribution pension plan selected an annuity payout.

Collectively, these low levels of voluntary annuitization stand in contrast to the implications of the Modigliani life-cycle hypothesis, as pointed out by Modigliani (1986). Indeed, as originally demonstrated by Yaari (1965) individuals with no utility of bequest should hold all their assets in actuarial notes, a.k.a. mortality-contingent annuities, since they stochastically dominate the payout from conventional asset classes. The result of Yaari (1965) has been the subject of much research in the public economics and insurance literature, and we refer the interested reader to a series of papers by Friedman and Warshawsky (1990), Brown (1999), Mitchell, Poterba, Warshawsky, and Brown (1999), Brown and Poterba (2000), and Brown and Warshawsky (2001). Collectively, these papers place some of the ‘blame’ for low annuitization rates on the high loads and fees that are embedded in annuity prices. Other economic-based explanations include Kotlikoff and Summers (1981), Kotlikoff and Spivak (1981), Hurd (1989), and Bernheim (1991), which focus on the role of families and their bequest motives on the demand for annuitization. Other models that focus on market imperfections and adverse selection include Brugiavini (1993) and Yagi and Nishigaki (1993).

Thus, given the rich literature on dynamic asset allocation and the increasing interest in pension-related finance issues, our objective is to incorporate longevity-insurance products into a portfolio and asset allocation framework that properly captures the actuarial and insurance imperfections. Although Richard (1975) extended Merton’s (1971) model to obtain Yaari’s (1965) results in a continuous-time framework, the institutional set-up lacked the realism of current payout annuity markets. Recent papers that attempt a portfolio-based model for annuitization along the same lines

– most written after the first draft of this paper was released – include Kapur and Orszag (1999), Blake, Cairns, and Dowd (2000), Neuberger (2003), Webb and Dushi (2003), Sinclair (2003), Stabile (2003), and Battocchio, Menoncin, and Scaillet (2003).

However, in addition to our objective of developing a comprehensive asset allocation model incorporating payout annuities, we also believe that the decision of when exactly to purchase an irreversible life annuity endows the holder with an incentive to delay which can be heuristically viewed as an option. Indeed, under many institutional pension arrangements (such as in the United Kingdom, or the rules regarding variable annuity saving policies in the U.S.) individuals are allowed to drawdown their pension via discretionary consumption but must eventually annuitize at *one point in time* their remaining wealth. We refer to this system as an all-or-nothing arrangement and argue that this is similar to Stock and Wise’s (1990) option to retire and echoes ideas by Stanton (2000) and Sundaresan and Zapatero (1997) who examine optimal behavior (and valuation) of various pension benefits. Other institutional structures allow for annuitization at any time and in small quantities as well, and we refer to these systems as slowly and anytime throughout the paper. We investigate the optimal annuitization policy in both of these cases, although the above-mentioned option value is only present in the former case.

3 Financial and Annuity Markets

In this section, we describe the general financial and annuity market prices. We assume that an individual can invest in a riskless asset whose price at time s , X_s , follows the process $dX_s = rX_s ds$, $X_t = x > 0$, for some fixed $r \geq 0$. Also, the individual can invest in a risky asset whose price at time s , S_s , follows geometric Brownian motion given by

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dB_s, \\ S_t = S > 0, \end{cases} \quad (1)$$

in which $\mu > r$, $\sigma > 0$, and B_s is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_s\}$ of the probability space $(\Omega, \mathcal{F}, \Pr)$. Let W_s be the wealth at time s of the individual, and let π_s be the amount that the decision maker invests in the risky asset at time s . Also, the decision maker consumes at a rate of c_s at time s . Then, the amount in the riskless asset is $W_s - \pi_s$, and when the individual buys no annuities, wealth follows the process

$$\left\{ \begin{array}{l} dW_s = d(W_s - \pi_s) + d\pi_s - c_s ds \\ \quad = r(W_s - \pi_s) dt + \pi_s (\mu ds + \sigma dB_s) - c_s ds \\ \quad = [rW_s + (\mu - r)\pi_s - c_s] ds + \sigma\pi_s dB_s, \\ W_t = w > 0. \end{array} \right. \quad (2)$$

In Sections 4 and 5 we assume that the decision maker seeks to maximize (over admissible $\{c_s, \pi_s\}$ and over times of annuitizing all his or her wealth, τ) the expected utility of discounted consumption. Admissible $\{c_s, \pi_s\}$ are those that are measurable with respect to the information available at time s , namely \mathcal{F}_s , that restrict consumption to be non-negative, and that result in (2) having a unique solution; see Karatzas and Shreve (1998). We also allow the individual to value expected utility via a subjective hazard rate (or force of mortality) while the annuity is priced by using an objective hazard rate.

Our financial economy is based on the (simpler) geometric Brownian motion plus risk-free rate model originally pioneered by Merton (1971), as opposed to the more recent and richer models developed by Kim and Omberg (1996), Sorensen (1999), or Wachter (2002), for example. The reason is that we are primarily interested in the implications of introducing a mortality contingent claim into the asset allocation framework, as opposed to studying the impact of stochastic interest rates or mean-reverting equity premiums *per se*. Thus, by avoiding and not paying the computational price of a more complex set-up, we are able to obtain analytical solutions to our annuitization problems.

We now move on to the insurance and actuarial assumptions. We let ${}_t p_x^S$ denote the subjective conditional probability that an individual aged (x) believes he or she will survive to age ($x + t$). It is defined via the subjective hazard function, λ_{x+t}^S , by the formula

$${}_t p_x^S = \exp\left(-\int_0^t \lambda_{x+s}^S ds\right). \quad (3)$$

We have a similar formula for the objective conditional probability of survival, ${}_t p_x^O$, in terms of the objective hazard function, λ_{x+t}^O . The actuarial present value of a life annuity that pays \$1 per year continuously to (x) is written \bar{a}_x . It is defined by

$$\bar{a}_x = \int_0^\infty e^{-rt} {}_t p_x dt. \quad (4)$$

We deliberately use the risk-free rate r in our annuity pricing because most of the recent empirical evidence suggests that the *money's worth* of annuities relative to the risk-free Government yield curve is relatively close to one. In other words the expected present value of payouts using the risk-free rate *is* equal to the premium paid for that benefit. Thus, it appears that the additional

credit risk that the insurance company might take-on by investing in higher risk bonds is offset by any insurance loads and commissions they charge. We refer the interested reader to the paper by Mitchell, Poterba, Brown and Warshawsky (1999) for a greater discussion of the precise curve that used for pricing in practice.

In terms of notation, if we use the subjective hazard rate to calculate the survival probabilities, then we write \bar{a}_x^S , while if we use the objective (pricing) hazard rate to calculate the survival probabilities, then we write \bar{a}_x^O . Just to clarify, by objective \bar{a}_x^O , we mean the actual market prices of the annuity net of any insurance loading, whereas \bar{a}_x^S denotes what the market price ‘would have been’ had the insurance company used the individual’s personal and subjective assessment of her mortality.

We refer the interested reader to Hurd and McGarry (1995, 1997) for a discussion of experiments involving “subjective” versus “objective” assessments of survival probabilities.¹ We will demonstrate that asymmetry of mortality beliefs might go a long way towards explaining why individuals who believe themselves to be less healthy than average are more likely to avoid annuities, despite having no declared bequest motive. In the classical *perfect market* Yaari (1965) framework, subjective survival rates do not play a role in the optimal policy. We will prove the result that as long as the consumer disagrees with the insurance company’s pricing basis regarding her subjective hazard rate - or personal health status - she will delay annuitization in an all-or-nothing environment.

In Section 6 we assume that the decision maker maximizes (over admissible $\{c_s, \pi_s, A_s\}$) the expected utility of discounted lifetime consumption and bequest, in which A_s is the annuity purchasing process. A_s denotes the non-negative annuity income *rate* at time s after any annuity purchases at that time; we assume that A_s is right-continuous with left limits. The source of this income could be previous annuity purchases or a preexisting annuity, such as Social Security or pension income. We assume that the individual can purchase an annuity at the price of \bar{a}_{x+s}^O per dollar of annuity income at time s , or equivalently, at age $x + s$. In that case, the dynamics of the wealth process is given by

$$\begin{cases} dW_s = [rW_{s-} + (\mu - r)\pi_s - c_s + A_{s-}] ds + \sigma\pi_s dB_s - \bar{a}_{x+s}^O dA_s, \\ W_{t-} = w > 0. \end{cases} \quad (5)$$

The negative sign on the subscripts for wealth and annuities denotes the left-hand limit of those quantities before any (lump-sum) annuity purchases.

¹Also, more recently, Smith, Taylor, and Sloan (2001) claim that their “findings leave little doubt that subjective perceptions of mortality should be taken seriously.” They state that individuals’ “longevity expectations are reasonably good predictions of future mortality.”

4 Restricted Market: All or Nothing

In this section we examine the institutional arrangement whereby the individual is required to annuitize all her wealth in a lump sum at some (retirement) time τ . If the volatility of investment return $\sigma = 0$, then we show that the individual annuitizes *all* her wealth at a time T for which $\mu = r + \lambda_{x+T}^O$, which is the time at which the mortality credits plus the risk free rate is equal to the expected return from the asset. Furthermore, if $\lambda_{x+t}^S = \lambda_{x+t}^O$ for all $t > 0$, then the individual will consume exactly the annuity income after time T . As an approximation to the case for which $\sigma > 0$, we assume that at some time τ , the individual annuitizes all her wealth W_τ and thereafter consumes at a rate of $\frac{W_\tau}{\bar{a}_{x+\tau}^O}$, the annuity income. It follows that the associated value function of this problem is given by

$$\begin{aligned}
 U(w, t) &= \sup_{\{c_s, \pi_s, \tau\}} E \left[\int_t^\tau e^{-r(s-t)} {}_{s-t} p_{x+t}^S u(c_s) ds + \int_\tau^\infty e^{-r(s-t)} {}_{s-t} p_{x+t}^S u\left(\frac{W_t}{\bar{a}_{x+\tau}^O}\right) ds \middle| W_t = w \right] \\
 &= \sup_{\{c_s, \pi_s, \tau\}} E \left[\int_t^\tau e^{-r(s-t)} {}_{s-t} p_{x+t}^S u(c_s) ds + e^{-r(\tau-t)} {}_{\tau-t} p_{x+t}^S u\left(\frac{W_\tau}{\bar{a}_{x+\tau}^O}\right) \bar{a}_{x+\tau}^S \middle| W_t = w \right] \quad (6)
 \end{aligned}$$

in which u is an increasing, concave utility function of consumption. Note that the individual discounts consumption at the riskless rate r . If we were to model with a subjective discount rate of say ρ , then this is equivalent to using r as in (6) and adding $\rho - r$ to the subjective hazard rate. Thus, there is no effective loss of generality in setting the subjective discount rate equal to the riskless rate r . Also, while some life-cycle models in the literature adjust the discount rate for perceived risk and other subjective factors, we remind the reader that our underlying hazard rate λ_{x+t}^S effectively adjusts the discount rate for the probability of survival and thus takes these risks into account implicitly.

Note that in this section we do not account for pre-existing annuities, from Social Security in the U.S for example. We anticipate that such annuities will change the optimal time of annuitization, but we defer this problem to Section 6. Also, note that we take the annuity prices as exogenously given. We are not creating an equilibrium (positive) model of pricing as in the adverse selection literature of Akerlof (1970) or Rothschild and Stiglitz (1976), but rather a normative model of how people should behave in the presence of these given market prices. Expanding to equilibrium considerations is beyond the scope of this (normative) paper.

We also restrict our attention to the case in which the utility function exhibits constant relative risk aversion (CRRA), $\gamma = -cu''(c)/u'(c)$. That is, u is given by

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \gamma > 0, \gamma \neq 1. \quad (7)$$

For this utility function, the relative risk aversion equals γ , a constant. The utility function that corresponds to relative risk aversion 1 is logarithmic utility. In Appendix A, we show that for CRRA utility, we can assume first that the optimal stopping annuitization time is some fixed time in the future, say T . Based on that value of T , we then find the optimal consumption and investment policies. Finally, we find the optimal value of $T \geq 0$.

To this end, define the value function V by:

$$V(w, t; T) = \sup_{\{c_s, \pi_x\}} E \left[\int_t^T e^{-r(s-t)} s_{-t} p_{x+t}^S \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-r(T-t)} T_{-t} p_{x+t}^S \frac{1}{1-\gamma} \left(\frac{W_T}{\bar{a}_{x+T}^S} \right)^{1-\gamma} \bar{a}_{x+T}^S | W_t = w \right]. \quad (8)$$

V solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases} (r + \lambda_{x+t}^S) V \\ = V_t + \max_{\pi} \left[\frac{1}{2} \sigma^2 \pi^2 V_{ww} + (\mu - r) \pi V_w \right] \\ + r w V_w + \max_{c \geq 0} \left[-c V_w + \frac{1}{1-\gamma} c^{1-\gamma} \right], \\ V(w, T; T) = \frac{1}{1-\gamma} \left(\frac{w}{\bar{a}_{x+T}^S} \right)^{1-\gamma} \bar{a}_{x+T}^S. \end{cases} \quad (9)$$

See Björk (1998) for clear derivations of such HJB equations. The optimal consumption and investment policies are given via the first-order conditions from (7) by

$$c^*(w, t) = (V_w(w, t))^{-\frac{1}{\gamma}} \quad (10)$$

and

$$\pi^*(w, t) = \frac{\mu - r}{\sigma^2} \frac{V_w(w, t)}{V_{ww}(w, t)}, \quad (11)$$

respectively. It is straightforward, but tedious, to show that for CRRA utility, V is given by

$$\begin{aligned} V(w, t; T) &= \frac{1}{1-\gamma} w^{1-\gamma} \left[\left\{ \frac{\bar{a}_{x+T}^S}{(\bar{a}_{x+T}^O)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} e^{-\frac{r-\delta(1-\gamma)}{\gamma}(T-t)} (T-t p_{x+t}^S)^{\frac{1}{\gamma}} + \int_t^T e^{-\frac{r-\delta(1-\gamma)}{\gamma}(s-t)} (s-t p_{x+t}^S)^{\frac{1}{\gamma}} ds \right]^{\gamma}, \end{aligned} \quad (12)$$

in which $\delta = r + \frac{(\mu-r)^2}{2\sigma^2\gamma}$. The optimal consumption and investment policies are given in feedback form by

$$C_t^* = c^*(W_t^*, t) = W_t^* k(t), \quad (13)$$

and

$$\Pi_t^* = \pi^*(W_t^*, t) = \frac{\mu - r}{\sigma^2 \gamma} W_t^*, \quad (14)$$

respectively, in which W_t^* is the optimally controlled wealth before annuitization (time T). Here, the function k is defined by

$$k^{-1}(t) = \left\{ \frac{\bar{a}_{x+T}^S}{(\bar{a}_{x+T}^O)^{1-\gamma}} \right\}^{\frac{1}{\gamma}} e^{-\frac{r-\delta(1-\gamma)}{\gamma}(T-t)} ({}_{T-t}p_{x+t}^S)^{\frac{1}{\gamma}} + \int_t^T e^{-\frac{r-\delta(1-\gamma)}{\gamma}(s-t)} ({}_{s-t}p_{x+t}^S)^{\frac{1}{\gamma}} ds \quad (15)$$

If we are in the case of logarithmic utility, as derived in detail by Milevsky and Young (2003a), then the optimal consumption rate is $C_t^* = \frac{W_t^*}{\bar{a}_{x+t}^S}$. It is interesting to note that if $r = 0$, in which case the denominator collapses to a (subjective) life expectancy, the consumption rate is precisely the minimum rate mandated by the Internal Revenue Service for annual consumption withdrawals from IRAs after age 71 in the U.S. Specifically, the proportion required to be withdrawn from one's annuity each year equals the start-of-year balance divided by the future expectation of life. Because $r > 0$ in reality, the minimum IRS-mandated consumption rate is less than what is optimal for individuals with logarithmic utility and with mortality equal to that in the IRS tables.

To find the optimal time of annuitization, differentiate V in (12) with respect to T . One can show that

$$\frac{\delta V}{\delta T} \propto \left[\frac{\gamma}{1-\gamma} \left(\frac{\bar{a}_{x+T}^S}{\bar{a}_{x+T}^O} \right)^{-\frac{1-\gamma}{\gamma}} - \frac{1}{1-\gamma} + \frac{\bar{a}_{x+T}^S}{\bar{a}_{x+T}^O} \right] + \bar{a}_{x+T}^S [\delta - (r + \lambda_{x+T}^O)]. \quad (16)$$

Thus, if the expression on the right-hand side of (16) is negative for all $T \geq 0$, then it is optimal to annuitize one's wealth immediately. However, if there exists a value $T^* > 0$ such that the right-hand side of (16) is positive for all $0 \leq T < T^*$ and is negative for all $T > T^*$, then it is optimal to annuitize one's wealth at time T^* . In all the examples we present below, one of these two conditions holds. The decision to annuitize is independent of one's wealth, an artifact of CRRA utility.

Note that if the subjective and objective forces of mortality are equal, then we have

$$\frac{\delta V}{\delta T} \propto [\delta - (r + \lambda_{x+T})]. \quad (17)$$

If the hazard rate λ_x is increasing with respect to age x , then *either* $\delta \leq (r + \lambda_x)$ from which it follows that it is optimal to annuitize one's wealth immediately, *or* $\delta > (r + \lambda_x)$, from which it

follows that there exists a time T in the future (possibly infinity) at which it is optimal to annuitize one's wealth. To interpret this result, recall that the rate of return (ignoring consumption) can be defined as: $r^a = 2\delta - r$; thus, the individual will annuitize his or her wealth when $r^a \leq r + 2\lambda_{x+T}$. It follows that annuitizing one's wealth is optimal as soon as the *excess* return, $r^a - r = \frac{(\mu-r)^2}{\sigma^2\gamma}$, is exceeded by *twice* the hazard rate. Stated differently, the optimal age to purchase a fixed immediate life annuity is as soon as the instantaneous force of mortality, λ_x , is greater than $\frac{(\mu-r)^2}{2\sigma^2\gamma}$. Thus, one can think of the hazard rate as a form of excess return on the annuity due to the fact that the wealth reverts to the insurance company when the buyer of the annuity dies.

In all our examples, we observe that if the subjective force of mortality is *different* than the objective force of mortality, then the optimal time of annuitization increases from the T given by the zero of the right-hand side of (16). We can show mathematically that this is true if the subjective force of mortality varies from the objective force to the extent that $\bar{a}_x^S < 2\bar{a}_x^O$ for all x (see Appendix B), and we conjecture that it is true in general. Note that this inequality is automatically true for people who are less healthy because in this case $\bar{a}_x^S < \bar{a}_x^O$ for all x . For an individual who is less healthy than the average person, the annuity will be too expensive, and the person will want to delay annuitizing her wealth. On the other hand, for an individual who is healthier than the average person, the annuity will be relatively cheap. However, such a healthy person will live longer on average and will be interested in receiving a larger annuity benefit by consuming less now and by waiting to buy the annuity later in life. Therefore, a healthy person is also willing to delay annuitizing her wealth in exchange for a larger annuity benefit (for a longer time).

Once again, we emphasize the counter-intuitive nature of this result. Regardless of whether the individual believes she is healthier or less healthy, compared to the mortality assumption used by the insurance company, she will delay annuitization. In the meantime, she will consume at the optimal rate - which will depend on her subjective mortality assessment - and then convert her accumulated savings to a life annuity at time T .

Of course, by following the optimal policies of investment, consumption, and annuitizing one's wealth, an individual runs the risk of being able to consume less after annuitizing wealth than if she had annuitized wealth immediately at time $t = 0$. Naturally, there is the chance of the exact opposite, namely that the lifetime annuity stream will be higher. Therefore, to quantify this risk, we calculate the probability associated with various consumption outcomes. See Appendix C for the formula of this probability. We include calculations of it in a numerical example below.

Finally, we define a metric for measuring the loss in value from annuitizing prematurely by

computing the additional wealth that would be required to compensate the utility maximizer for forced annuitization. This is akin to the annuity equivalent wealth used by MPWB (1999), which we prefer to label a subjective option value. Technically, it is defined to be the least amount of money h that when added to current wealth w makes the person indifferent between annuitizing now (with the extra wealth) and annuitizing at time T (without the extra wealth). Thus, h is given by

$$V(w, t; T) = V(w + h, t; 0), \tag{18}$$

in which T is the optimal time of annuitization and V is given by (12). In the examples in the next section, we express h as a percentage of wealth w . This is appropriate because V exhibits CRRA with respect to w .

5 Numerical Examples

In this section, we present two numerical examples to illustrate the results from the previous section. To start, although most mortality tables are discretized, we require a continuous-time mortality law. We therefore use a Gompertz force of mortality, which is common in the actuarial literature for annuity pricing; see Frees, Carriere, and Valdez (1966). This model for mortality has also been employed in the economics literature for pricing insurance; see Johansson (1996), for example. The force of mortality is written $\lambda_x = \exp((x - m)/b)/b$ in which m is a modal value and b is a scale parameter. Note that the force of mortality increases exponentially with age. In this paper we fit the parameters of the Gompertz, namely m and b , to the Individual Annuity Mortality 2000 (basic) Table with projection scale G. For males, we fit parameters $(m, b) = (88.18, 10.5)$; for females, $(92.63, 8.78)$. Initially, we assume that the subjective and objective forces of mortality are equal. Throughout this section we assume that the seller of the annuity uses the female hazard rate to price annuities for women; similarly, for men. Figure 1 shows the graph of the probability density function (PDF) of the future-lifetime random variable under a Gompertz hazard rate that is fitted to the discrete mortality table.²

Figure 1 about here.

²We actually fit a Makeham hazard rate, or force of mortality, namely $\lambda + \exp((x - m)/b)/b$ in which $\lambda \geq 0$ is a constant that models an accident rate. However, the fitted value of λ was 0, so the effective form of the hazard rate is Gompertz (Bowers et al., 1977).

As for the capital market parameters, in both our examples, the risky stock is assumed to have drift $\mu = 0.12$ and volatility $\sigma = 0.20$. This is roughly in line with Ibbotson Associates (2001) numbers, which are widely used by practitioners when simulating long-term investment returns. We assume that the nominal rate of return of the riskless bond is $r = 0.06$. We display values for the option to delay annuitization h , for two different levels of risk aversion, $\gamma = 1$ (logarithmic utility) and $\gamma = 2$. A variety of studies have estimated the value of γ to lie between 1 and 2. See, the paper by Friend and Blume (1975) that provides an empirical justification for constant relative risk aversion, as well as the more recent MPWB (1999) paper in which the CRRA value is taken between 1 and 2. In the context of estimating the present value of a variable annuity for Social Security, Feldstein and Rangelova (2001) provide some qualitative arguments that the value of CRRA is less than 3 and probably even less than 2.

5.1 Example #1

Table 1 provides the optimal age of annuitization – and what we have labeled the value of the option to delay as a percentage of initial wealth – as well as the probability of consuming less at the optimal time of annuitization than if one had annuitized one’s wealth immediately. We refer to this as the probability of deferral failure.

Table 1 about here.

Note that females wish to annuitize at older ages compared to males because the mortality rate of females is lower at each given age. Also, note that more risk averse individuals wish to annuitize sooner, an intuitively pleasing result. Finally, our pedagogically appealing value of the option to delay annuitization – which is effectively the certainly equivalent of the welfare loss from annuitizing immediately – decreases as one gets closer to the optimal age of annuitization, as one expects.

The probability of deferral failure, although seemingly high, is balanced by the probability of ending up with more than, say, 20% of the original annuity amount. For example, for a 70-year-old female with $\gamma = 2$, the probability of consuming at least 20% more at the optimal age of annuitization than if she were to annuitize immediately is 0.474. Obviously, on a utility-adjusted basis this is a worthwhile trade-off as evidenced by the behavior of the value function. See Table 2 for tabulations of the probability that the individual consumes at least 20% more at the optimal age of annuitization than if he or she were to annuitize immediately, for various ages and for $\gamma = 1$ and 2.

Table 2 about here.

These “upside” probabilities decrease as the optimal age of annuitization approaches. Also, for a given age, they decrease as the CRRA increases. This makes sense because a less risk-averse person is less willing to face a distribution with a higher variance.

5.2 Example #2

Continue the assumptions in the previous example as to the financial market. Suppose that we have a male aged 60 with $\gamma = 2$, whose objective mortality follows that from the previous example; that is, annuity prices are determined based on the hazard rate given there. For this example, suppose that the subjective force of mortality is a multiple of the objective force of mortality; specifically, $\lambda_x^S = (1 + f)\lambda_x^O$, in which f ranges from -1 (immortal) to infinity (at death’s door). This transformation is called the *proportional hazard* transformation in actuarial science introduced by Wang (1996), and it is similar to the transformation examined by Johansson (1996) in the economic context of the value of increasing one’s life expectancy.

In Table 3 we present the imputed value of the option to delay annuitization, the optimal age of annuitization, the optimal rate of consumption before annuitization (as a percentage of current wealth), and the rate of consumption after annuitization (also, as a percentage of current wealth). For comparison, if the male were to annuitize his wealth at age 60, the rate of consumption would be 8.34%. Also, the optimal proportion invested in the risky stock before annuitization is 75%.

Table 3 about here.

Note that as the 60-year-old male’s subjective mortality gets closer to the objective (pricing) mortality, then the optimal age of annuitization decreases. It seems that the optimal age of annuitization will be a minimum when the subjective and objective forces of mortality equal, at least for increasing forces of mortality. We conjecture that this result is true in general, but we only have a proof of it when $\bar{a}_x^S < 2\bar{a}_x^O$; see Appendix B. Also, note that the consumption rate before annuitization increases as the person becomes less healthy, as expected.

Compare these rates of consumption with 8.34%, the rate of consumption if the male were to annuitize his wealth immediately. We see that if the male is healthy relative to the pricing force of mortality, then he is willing to forego current consumption in exchange for greater consumption when he annuitizes, at least up to $f = -0.4$. Past that point, the optimal rate of consumption before annuitization is greater than 8.34%. For a 60-year-old male with $f = -0.2$ (20% more healthy than

average), see Figure 2 for a graph of the expected consumption rate as a percentage of *initial* wealth. We also graph the 25th and 75th percentiles of his random consumption. This individual expects to live to age 84.4. Note that the annuitant has roughly a 70% chance of consuming more throughout the remaining life compared to annuitizing at age 60.

Figure 2 about here.

6 Open Market: Slowly and Anytime

In this section, we consider the optimal annuity-purchasing problem for an individual who seeks to maximize her expected utility of lifetime consumption and bequest. In the first subsection, we allow the individual to have rather general preferences, while in the second subsection we specialize to the case for which preferences exhibit constant relative risk aversion. We allow the individual to buy annuities in lump sums or continuously, whichever is optimal. Our results are similar to those of Dixit and Pindyck (1994, pp 359ff). They consider the problem of a firm's irreversible capacity expansion. For our individual, annuity purchases are also irreversible, and this leads to the similarity in results.

In the third subsection we linearize the HJB equation in the region of no-annuity purchasing via a convex dual transformation in the case for which there is no bequest motive. In the fourth subsection we provide an implicit analytical solution to the optimal annuity purchasing problem in the case for which the force of mortality is constant. We end with an example demonstrating our method.

6.1 General Utility of Consumption and Bequest

In this section we show that the individual's optimal annuity purchasing is given by a barrier policy in that she will annuitize just enough of her wealth to stay on one side of the barrier in wealth-annuity space. In equation (5), we describe the dynamics of the wealth for this individual. Denote the random time of death of our individual by τ_d . Thus, her value function at time t , or at age $x + t$, is given by

$$\begin{aligned}
 U(w, A, t) &= \sup_{\{c_s, \pi_s, A_{s-}\}} E \left[\int_t^\infty e^{-r(s-t)} {}_{s-t}p_{x+t}^S u_1(c_s) ds + e^{-r(\tau_d-t)} u_2(W_{\tau_d}) | W_{t-} = w, A_{t-} = A \right] \\
 &= \sup_{\{c_s, \pi_s, A_{s-}\}} E \left[\int_t^\infty e^{-r(s-t)} {}_{s-t}p_{x+t}^S \{u_1(c_s) + \lambda_{x+s}^S u_2(W_{s-})\} ds | W_{t-} = w, A_{t-} = A \right] \quad (19)
 \end{aligned}$$

in which u_1 and u_2 are strictly increasing, concave utility functions of consumption and bequest, respectively. Note that we assume the individual discounts future consumption at the riskless rate r since the mortality discounting – which increases the effective discount rate – is incorporated separately. The value function U is jointly concave in w and A . We refer the interested reader to the earlier working paper version by Milevsky and Young (2002) for a detailed discussion of this and other properties of U .

We continue with a formal discussion of the derivation of the associated HJB equation. Suppose that at the point (w, A, t) , it is optimal *not* to purchase any annuities. It follows from Itô's lemma that U satisfies the equation at (w, A, t) given by

$$\begin{aligned} & (r + \lambda_{x+t}^S)U \\ &= U_t + (rw + A)U_w + \max_{\pi} \left[\frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r) \pi U_w \right] + \max_{c \geq 0} [-cU_w + u_1(c)] + \lambda_{x+t}^S u_2(w). \end{aligned} \quad (20)$$

Because the above policy is in general suboptimal, (20) holds as an inequality; that is, for all (w, A, t) ,

$$\begin{aligned} & (r + \lambda_{x+t}^S)U \\ & \geq U_t + (rw + A)U_w + \max_{\pi} \left[\frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r) \pi U_w \right] + \max_{c \geq 0} [-cU_w + u_1(c)] + \lambda_{x+t}^S u_2(w). \end{aligned} \quad (21)$$

Next, assume that at the point (w, A, t) it is optimal to buy an annuity instantaneously. In other words, assume that the investor moves instantly from (w, A, t) to $(w - \bar{a}_{x+t}^O \Delta A, A + \Delta A, t)$. Then, the optimality of this decision implies that

$$U(w, A, t) = U(w - \bar{a}_{x+t}^O \Delta A, A + \Delta A, t), \quad (22)$$

which in turns yields

$$U_A(w, A, t) - \bar{a}_{x+t}^O U_w(w, A, t) = 0. \quad (23)$$

Note that the lump-sum purchase is such that the marginal utility of annuity income equals the adjusted marginal utility of wealth, in which we adjust the marginal utility of wealth by multiplying by the cost of \$1 of annuity income. This result parallels many such in economics. Indeed, the

marginal utility of annuity income can be thought of as the marginal utility of the benefit, while the adjusted marginal utility of wealth can be thought of as the marginal utility of the cost. Thus, the lump-sum purchase is such that the marginal utilities of benefit and cost are equal.

However, such a policy is in general suboptimal; therefore, (22) holds as an inequality and (23) becomes

$$\bar{a}_{x+t}^O U_w(w, A, t) - U_A(w, A, t) \geq 0. \quad (24)$$

By combining (21) and (24), we obtain the HJB equation (25) below associated with the value function U given in (19). The following result can be proved as in Zariphopoulou (1992), for example.

Proposition 6.1: *The value function U is a constrained viscosity solution of the Hamilton-Jacobi-Belman equation*

$$\begin{aligned} \min & \left[(r + \lambda_{x+t}^S)U - U_t - (rw + A)U_w - \max_{\pi} \left(\frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r)\pi U_w \right) \right. \\ & \left. - \max_{c \geq 0} (-cU_w + u_1(c)) - \lambda_{x+t}^S u_2(w), \bar{a}_{x+t}^O U_w - U_A \right] = 0. \end{aligned} \quad (25)$$

Equation (23) defines a “barrier” in wealth-annuity income space. If wealth and annuity income lie to the right of the barrier at time t , then the individual will immediately spend a lump sum of wealth to move diagonally to the barrier (up and to the left). The move is diagonal because as wealth decreases to purchase more annuities, annuity income increases. Thereafter, annuity income is either constant if wealth is low enough to keep to the left of the barrier, or annuity income responds continuously to infinitesimally small changes of wealth at the barrier.

Thus, as in Dixit and Pindyck (1994, pp 359ff) or in Zariphopoulou (1992), we have discovered that the optimal annuity-purchasing scheme is a type of barrier control. Other barrier control policies appear in finance and insurance. In finance, Zariphopoulou (1999, 2001) reviews the role of barrier policies in optimal investment in the presence of transaction costs; also see the references within her two articles. See Gerber (1979) for a classic text on risk theory in which he includes a section on optimal dividend payout and shows that it follows a type of barrier control.

6.2 Constant Relative Risk Aversion Preferences

In this subsection we specialize the results of the previous subsection to the case for which the individual’s preferences exhibit CRRA. For this case, we can reduce the problem by one dimension, and we show that the barrier given in the previous section is a ray emanating from the origin in wealth-annuity space. Let

$$u_1(c) = \frac{1}{1-\gamma}c^{1-\gamma}, \text{ and } u_2(w) = ku_1(w), \quad \gamma > 0, \gamma \neq 1, k \geq 0. \quad (26)$$

The parameter $k \geq 0$ weights the utility of bequest relative to the utility of consumption. Davis and Norman (1990) and Shreve and Soner (1994) show that for CRRA preferences in the problem of consumption and investment in the presence of transaction costs, the value function U is a solution of its HJB equation in the classical sense, not just in the viscosity sense. Generally, if the force of mortality is “eventually” large enough to make the value function well-defined, then this result holds for our problem, too.

For the utility functions in (26), it turns out that the value function U is homogeneous of degree $1 - \gamma$ with respect to wealth w and annuity income A . That is, $U(bw, bA, t) = b^{1-\gamma}U(w, A, t)$ for $b > 0$. Thus, if we define V by $V(z, t) = U(z, 1, t)$, then we can recover U from V by

$$U(w, A, t) = A^{1-\gamma}V(w/A, t), \text{ for } A > 0. \quad (27)$$

It follows that the HJB equation for U from Proposition 6.1 becomes the following equation for V :

$$\begin{aligned} \min \left[(r + \lambda_{x+t}^S)V - V_t - (rz + 1)V_z - \max_{\hat{\pi}} \left(\frac{1}{2}\sigma^2\hat{\pi}^2V_{zz} + (\mu - r)\hat{\pi}V_z \right) \right. \\ \left. - \max_{\hat{c} \geq 0} \left(-\hat{c}V_z + \frac{\hat{c}^{1-\gamma}}{1-\gamma} \right) - k\lambda_{x+t}^S \frac{z^{1-\gamma}}{1-\gamma}, \quad (z + \bar{a}_{x+t}^O)V_z - (1 - \gamma)V \right] = 0, \end{aligned} \quad (28)$$

in which $\hat{c} = c/A$, and $\hat{\pi} = \pi/A$. Davis and Norman (1990) and Shreve and Soner (1994) use the same transformation in the problem of consumption and investment in the presence of transaction costs. Also, Duffie and Zariphopoulou (1997) and Koo (1998) use this transformation to study optimal consumption and investment with stochastic income.

Milevsky and Young (2002) study properties of the optimal consumption and investment policies. Please refer to that work for details on the proof of the following proposition that describes the actions of the individual.

Proposition 6.2: *For each value of $t \geq 0$, there exists a value of the wealth-to-income ratio $z_0(t)$ that solves*

$$(z_0(t) + \bar{a}_{x+t}^O)V_z(z_0(t), t) = (1 - \gamma)V(z_0(t), t), \quad (29)$$

such that

- (i) *If $z = w/A > z_0(t)$, then the individual immediately buys an annuity so that*

$$\frac{w - \Delta A \bar{a}_{x+t}^O}{A + \Delta A} = z_0(t); \quad (30)$$

Thus, $V(z, t) = V(z_0(t), t)$ in this case.

(ii) If $z = w/A < z_0(t)$, then the individual buys no annuity; i.e., she is in the region of inaction. Thus, in this case, V solves the p.d.e. given by

$$\begin{aligned} & (r + \lambda_{x+t}^S)V \\ & = V_t + (rz + 1)V_z + \max_{\hat{\pi}} \left(\frac{1}{2} \sigma^2 \hat{\pi}^2 V_{zz} + (\mu - r) \hat{\pi} V_z \right) + \max_{\hat{c} \geq 0} \left(-\hat{c} V_z + \frac{\hat{c}^{1-\gamma}}{1-\gamma} \right) + k \lambda_{x+t}^S \frac{z^{1-\gamma}}{1-\gamma}. \end{aligned} \quad (31)$$

It follows that at each time point, the barrier $w = z_0(t)A$ is a ray emanating from the origin and lying in the first quadrant of (w, A) space.

Note that if $z_0(t) < \infty$, then it is optimal for the individual to have *positive* annuity income because the positive w axis lies in the region $\{(w, A, t) : w/A < z_0(t)\}$.

Davis and Norman (1990) and Shreve and Soner (1994) find results similar to those in Proposition 6.2 for the problem of optimal consumption and investment in the presence of proportional transaction costs. In the next subsection we show how to linearize the HJB equation of the individual who has no bequest motive.

6.3 Zero Bequest Motive: Linearization of the HJB Equation

In this subsection we linearize the nonlinear partial differential equation for V in the region of inaction given by equation (31) with no bequest motive ($k = 0$). To do this, we consider the convex dual of V defined by

$$\tilde{V}(y, t) = \max_{z > 0} [V(z, t) - zy]. \quad (32)$$

The critical value z^* solves the equation $0 = V_z(z, t) - y$; thus, $z^* = I(y, t)$, in which I is the inverse of V_z with respect to z . It follows that

$$\tilde{V}(y, t) = V[I(y, t), t] - yI(y, t). \quad (33)$$

Note that

$$\begin{aligned}
\tilde{V}_y(y, t) &= V_z[I(y, t)]I_y(y, t) - I(y, t) - yI_y(y, t) \\
&= yI_y(y, t) - I(y, t) - yI_y(y, t) \\
&= -I(y, t).
\end{aligned} \tag{34}$$

We can retrieve the function V from \tilde{V} by the relationship

$$V(z, t) = \min_{y>0} [\tilde{V}(y, t) + zy]. \tag{35}$$

Indeed, the critical value y^* solves the equation $0 = \tilde{V}_y(y, t) + z = -I(y, t) + z$; thus, $y^* = V_z(z, t)$, and

$$\begin{aligned}
\tilde{V}(y^*, t) + zy^* &= \tilde{V}[V_z(z, t), t] + zV_z(z, t) \\
&= V[I(V_z(z, t), t)] - V_z(z, t)I(V_z(z, t), t) + zV_z(z, t) \\
&= V(z, t) - zV_z(z, t) + zV_z(z, t) \\
&= V(z, t),
\end{aligned} \tag{36}$$

in which we use equation (33) for the second equality.

Next, note that

$$\tilde{V}_{yy}(y, t) = -I_y(y, t) = -1/V_{zz}[I(y, t), t], \tag{37}$$

and

$$\begin{aligned}
\tilde{V}_t(y, t) &= V_z[I(y, t), t]I_t(y, t) + V_t[I(y, t), t] - yI_t(y, t) \\
&= yI_t(y, t) + V_t[I(y, t), t] - yI_t(y, t) \\
&= V_t[I(y, t), t].
\end{aligned} \tag{38}$$

In the partial differential equation for V with no bequest motive ($k = 0$), let $z = I(y, t)$ and rewrite the equation in terms of \tilde{V} to obtain

$$\tilde{V}_t - (r + \lambda_{x+t}^S)\tilde{V} + \lambda_{x+t}^S y \tilde{V}_y + m y^2 \tilde{V}_{yy} = -y - \frac{\gamma}{1-\gamma} y^{1-\frac{1}{\gamma}}, \tag{39}$$

in which $m = \frac{1}{2} \left(\frac{\mu-r}{\sigma} \right)^2$. Note that (39) is a *linear* partial differential equation.

Next, consider the boundary condition $U_A(w, A, t) = \bar{a}_{x+t}^O U_w(w, A, t)$ from equation (23). In terms of V , this condition can be written as in equation (27), and we repeat it here for convenience

$$-(1 - \gamma)V(z_0(t), t) + (z_0(t) + \bar{a}_{x+t}^O)V_z(z_0(t), t) = 0. \quad (40)$$

Smooth pasting at the boundary implies that the derivative of this boundary condition with respect to z evaluated at $z = z_0(t)$ holds and is given by

$$\gamma V_z(z_0(t), t) + (z_0(t) + \bar{a}_{x+t}^O)V_{zz}(z_0(t), t) = 0. \quad (41)$$

We also have a boundary condition at $z = 0$ because at that point, the individual has no wealth to invest in the risky asset. Write $\hat{\pi}^*$ in terms of \tilde{V} : $\hat{\pi}^*(y, t) = \frac{\mu-r}{\sigma^2} y \tilde{V}_{yy}$. Thus, for $z = 0$ (with the corresponding value for y written $y_a(t)$), we have that either $y_a(t) = 0$ or $\tilde{V}_{yy}(y_a(t), t) = 0$.

Because $V_z > 0$ is strictly decreasing with respect to z , we have $y_a(t) > y_0(t) \geq 0$ for all $t \geq 0$, in which $y_a(t)$ and $y_0(t)$ are defined by

$$y_a(t) = V_z(0, t), \quad (42)$$

and

$$y_0(t) = V_z(z_0(t), t). \quad (43)$$

Thus, because $y_a(t) > 0$, in terms of \tilde{V} , the boundary conditions become

$$\tilde{V}_y(y_a(t), t) = 0, \quad (44)$$

for

$$\tilde{V}_{yy}(y_a(t), t) = 0, \quad (45)$$

and

$$(1 - \gamma)\tilde{V}(y_0(t), t) + \gamma y_0(t)\tilde{V}_y(y_0(t), t) = \bar{a}_{x+t}^O y_0(t), \quad (46)$$

for

$$\tilde{V}_y(y_0(t), t) + \gamma y_0(t)\tilde{V}_{yy}(y_0(t), t) = \bar{a}_{x+t}^O. \quad (47)$$

To solve the second-order partial differential equation (39) with free boundaries determined by (44 - 47), we propose the following algorithm. First, suppose we have estimates of the functions y_a and y_0 . Use these estimates to solve the partial differential equation (39) with conditions at y_a and y_0 given by equations (44) and (46). Re-estimate y_a and y_0 by using equations (45) and (47), and repeat the process until it converges. Implementing this algorithm is the subject of future research. In the next subsection, we solve the system in the case for which the forces of mortality are constant.

6.4 Constant Force of Mortality

If we assume that the forces of mortality are constant, that is, $\lambda_{x+t}^S \equiv \lambda^S$ and $\lambda_{x+t}^O \equiv \lambda^O$ for all $t \geq 0$, then we can obtain an “implicit” analytical solution of the value function V via the boundary-value problem given by (39) and (44 - 47). See Neuberger (2003) for recent and related work. In this case, V , \tilde{V} , y_a , and y_0 are independent of time, so (39) becomes the ordinary differential equation

$$-(r + \lambda^S)\tilde{V}(y) + \lambda^S y \tilde{V}'(y) + m y^2 \tilde{V}''(y) = -y - \frac{\gamma}{1 - \gamma} y^{1 - \frac{1}{\gamma}}, \quad (48)$$

with boundary conditions

$$\tilde{V}''(y_a) = 0, \quad (49)$$

for

$$\tilde{V}'(y_a) = 0, \quad (50)$$

and

$$(1 - \gamma)\tilde{V}(y_0) + \gamma y_0 \tilde{V}'(y_0) = \frac{y_0}{r + \lambda^O}, \quad (51)$$

for

$$\tilde{V}'(y_0) + \gamma y_0 \tilde{V}''(y_0) = \frac{1}{r + \lambda^O}. \quad (52)$$

The general solution of (48) is

$$\tilde{V}(y) = D_1 y^{B_1} + D_2 y^{B_2} + \frac{y}{r} + C_2 y^{1 - \frac{1}{\gamma}}, \quad (53)$$

with D_1 and D_2 constants to be determined by the boundary conditions, with C_2 given by

$$C_2 = r + \frac{\lambda^S}{\gamma} - m \frac{1-\gamma}{\gamma^2}, \quad (54)$$

with B_1 and B_2 given by

$$B_1 = \frac{1}{2m} \left[(m - \lambda^S) + \sqrt{(m - \lambda^S)^2 + 4m(r + \lambda^S)} \right] > 1, \quad (55)$$

and

$$B_2 = \frac{1}{2m} \left[(m - \lambda^S) - \sqrt{(m - \lambda^S)^2 + 4m(r + \lambda^S)} \right] < 0. \quad (56)$$

The boundary conditions at y_0 give us

$$D_1 \{1 + \gamma(B_1 - 1)\} y_0^{B_1} + D_2 \{1 + \gamma(B_2 - 1)\} y_0^{B_2} + \frac{y_0}{r} = \frac{y_0}{r + \lambda^O}, \quad (57)$$

and

$$D_1 B_1 \{1 + \gamma(B_1 - 1)\} y_0^{B_1} + D_2 B_2 \{1 + \gamma(B_2 - 1)\} y_0^{B_2} + \frac{y_0}{r} = \frac{y_0}{r + \lambda^O}. \quad (58)$$

Solve equations (57) and (58) to get D_1 and D_2 in terms of y_0 :

$$D_1 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{1 - B_2}{B_1 - B_2} \frac{y_0^{1-B_1}}{1 + \gamma(B_1 - 1)}, \quad (59)$$

and

$$D_2 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1 - 1}{B_1 - B_2} \frac{y_0^{1-B_2}}{1 + \gamma(B_2 - 1)}. \quad (60)$$

Next, substitute for D_1 and D_2 in $\tilde{V}'(y_a) + \gamma y_a \tilde{V}''(y_a) = 0$ from (49) and (50) to get

$$\frac{\lambda^O}{r + \lambda^O} \frac{B_1(1 - B_2)}{B_1 - B_2} \left(\frac{y_a}{y_0}\right)^{B_1-1} + \frac{\lambda^O}{r + \lambda^O} \frac{B_2(B_1 - 1)}{B_1 - B_2} \left(\frac{y_a}{y_0}\right)^{B_2-1} = 1. \quad (61)$$

(61) gives us an equation for the ratio $y_a/y_0 > 1$. To check that (61) has a unique solution greater than 1, note that the left-hand side (i) equals $\lambda^O/(r + \lambda^O) < 1$ when we set $y_a/y_0 = 1$, (ii) goes to infinity as y_a/y_0 goes to infinity, and (iii) is strictly increasing with respect to y_a/y_0 .

Next, substitute for D_1 and D_2 in $\tilde{V}'(y_a) = 0$ from (49) to get

$$\begin{aligned} -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1(1 - B_2)}{B_1 - B_2} \frac{(y_a/y_0)^{B_1-1}}{1 + \gamma(B_1 - 1)} - \frac{\lambda^O}{r(r + \lambda^O)} \frac{B_2(B_1 - 1)}{B_1 - B_2} \frac{(y_a/y_0)^{B_2-1}}{1 + \gamma(B_2 - 1)} \\ + \frac{1}{r} + C_2 \left(1 - \frac{1}{\gamma}\right) y_a^{-\frac{1}{\gamma}} = 0. \end{aligned} \quad (62)$$

Substitute for y_a/y_0 in equation (62), and solve for y_a . Finally, we can get y_0 from

$$y_0 = \frac{y_a}{y_a/y_0}, \quad (63)$$

and D_1 and D_2 from equations (59) and (60), respectively.

Once we have the solution for \tilde{V} , we can recover V from

$$\begin{aligned} V(z) &= \max_{y>0} [\tilde{V}(y) + zy] \\ &= \max_{y>0} \left[D_1 y^{B_1} + D_2 y^{B_2} + \frac{y}{r} + C_2 y^{1-\frac{1}{\gamma}} + zy \right], \end{aligned} \quad (64)$$

in which the critical value y^* solves

$$D_1 B_1 y^{B_1-1} + D_2 B_2 y^{B_2-1} + \frac{1}{r} + C_2 \left(1 - \frac{1}{\gamma} \right) y^{-\frac{1}{\gamma}} + z = 0. \quad (65)$$

Thus, for a given value of $z = w/A$, solve (65) for y and plug that value of y into (64) to get $U(w, A) = V(z)$. Perhaps more importantly, we are interested in the critical value z_0 above which an individual spends a lump-sum to purchase more annuity income. We pursue this in the following example.

6.5 Numerical Example

Suppose we have the following values for the hazard rate parameters. $\lambda^S = \lambda^O = 0.04$. That is, the force of mortality is constant and therefore the expected future lifetime is $1/\lambda = 25$ years. Furthermore, we set the risk-free interest rate to be $r = 0.04$, the expected return from the risky asset is $\mu = 0.08$ and the investment volatility is $\sigma = 0.20$. We have selected these numbers – which are lower than those used in the earlier examples – to better capture a real (after inflation) case in which social security benefits would be considered as part of the pre-existing annuity.

In Table 4, for various values of γ , we give the critical value of the ratio of wealth to annuity income $z_0 = w/A$ above which the individual will spend a lump-sum of wealth to increase her annuity income. We also include the amount that the individual will spend on annuities for a given (pre-existing) annuity income of $A = \$25,000$, namely $(w - z_0 A)/(1 + (r + \lambda^O)z_0)$.

Table 4 about here.

Notice from Table 4 that the amount spent on annuities increases for a given level of wealth as the individual becomes more risk averse, an intuitively pleasing result. Also, for a given level of risk aversion, the amount spent on annuities decreases as wealth decreases. In Table 5 we present the results for the case when $A = \$50,000$.

Table 5 about here.

We emphasize that these numerical results are appropriate when the forces of mortality are constant and when there is no bequest motive. To eliminate both of these restrictions is the subject of ongoing research.

7 Conclusion and Main Insights

Our paper uses the techniques and models of continuous-time financial economics introduced by Merton (1971) to locate the optimal behavior of a utility-maximizing individual who is interested in incorporating lifetime payout annuities (i.e. pensions) into his or her retirement portfolio. We have investigated a variety of institutional arrangements and market structures with differing restrictions and constraints. Our main results can be stated as follows.

- We argue that an individual who is faced with the irreversible decision of when to start a fixed (nominal or real) lifetime pension annuity – with the proviso that annuitization must take place in an ‘all or nothing’ format – is endowed with an incentive to delay that is substantially valuable at younger ages.
- In our model there is an incentive to delay annuitization even in the absence of bequest motives – which is in contrast to the classical Yaari (1965) result which has recently been extended by Davidoff, Brown and Diamond (2003) – because of the market imperfections that do not allow retirees to purchase payout annuities that offer complete asset allocation and flexibility to match ones subjective consumption preferences *vis a vis* their health status. Stated differently, a market in which instantaneously renegotiated life-contingent tontines are available would not give rise to our ‘option to delay’ result.
- Under this ‘all or nothing’ framework which is at the heart of many global public and private pension systems, the optimal age to annuitize is the age at which the ‘option to delay’ has zero value. This value is defined equal to the loss in utility that comes from not being able to behave optimally and, therefore, depends on a person’s coefficient of relative risk aversion as well as their subjective health status.
- Using historical market parameters and realistic mortality estimates, we conclude that in this ‘all or nothing’ framework the optimal age to purchase a pure life-contingent annuity does not occur prior to age 70. This result is consistent with a variety of probabilistic-based models which are based on the relationship between mortality credits and the returns from competing

asset classes. This after-age-70 result has also been advocated in a variety of popular-press article, such as a recent piece in the *Wall Street Journal*.³

- In the event that complete asset allocation flexibility is available within the payout annuity, which is akin to variable immediate annuities which are available in some countries and jurisdictions, the optimal age to annuitize is indeed earlier, although high fees and expenses relative to non-annuitized wealth can have a strong mitigating impact on the benefits from annuitization. Of course, bequest motives will impact this as well.
- When we move towards an open institutional system in which annuitization can take place in small portions and at anytime, we find that utility-maximizing investors should acquire a base amount of annuity income (i.e. social security or a defined benefit pension) and then annuitize additional amounts if and when their wealth-to-income ratio exceeds a certain level. In this case, individuals annuitize a fraction of wealth as soon as they have opportunity to do so – i.e. they do *not* wait – and they then purchase more annuities as they become wealthier.
- Thus, in contrast to the all-or-nothing pension structure, in the case of an open system where annuities can be purchased on an ongoing basis, we find that individuals prior to age 70 *should* have a minimal amount of annuity income and should immediately annuitize a fraction of wealth to create this base level of lifetime income if they do not already have this from pre-existing defined benefit pensions. We reiterate that individuals should *always* hold some annuities, even in the presence of a bequest motive, as long as $z_0(t)$ in Proposition 6.2 is less than infinity.

Thus, our paper extends and merges some of the classical results derived by Yaari (1965) into a multi-period asset allocation framework in which the real-world pension and institutional restrictions are explicitly modeled.

In terms of further research, most of the discussion and numerical analysis in this paper has centered around a model with no bequest motives, which is somewhat unrealistic given the large amounts of inheritance that we observed being bequeathed in practice. To this end, the authors are currently working on studying the impact of bequest motives on the optimal timing of, and demand for, annuity purchases. Likewise, the authors are looking at optimal payout annuity design that would increase the appeal of annuitization, something which is critically important given the exceedingly low levels of voluntary annuitization observed in practice.

³*Wall Street Journal*, 3 September 2003, page C1, “It Pays to Delay: The Longer You Wait To Buy an Annuity, the More You Get,” by Jonathan Clements.

8 Appendix

8.1 Appendix A: Separation Theorem

In this appendix we verify the approach in Section 4 for CRRA utility, for which we assume first that the optimal stopping time is some fixed time in the future, say T . Based on that value of T , we then find the optimal consumption and investment policies. Finally, we find the optimal value of $T \geq 0$.

From Krylov (1980, p. 13), the HJB equation for U from (6) is

$$g - U + \sup_{c, \pi} \left[(rw + (\mu - r)\pi - c)U_w + \frac{1}{2}\sigma^2 U_{ww} + U_t - (r + \lambda_{x+t}^S)U + f + U - g \right] = 0, \quad (66)$$

in which

$$f = u(c) \text{ and } g = \bar{a}_{x+t}^s u\left(\frac{w}{\bar{a}_{x+t}^O}\right). \quad (67)$$

Thus,

$$(r + \lambda_{x+t}^S)U \geq U_t + rwU_w + \max_c [u(c) - cU_w] + \max_\pi \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2 \pi^2 U_{ww} \right], \quad (68)$$

with equality if

$$U(w, t) > \bar{a}_{x+t}^s u\left(\frac{w}{\bar{a}_{x+t}^O}\right). \quad (69)$$

Now, for $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$, we have $U(w, t) = \frac{1}{1-\gamma}w^{1-\gamma}\psi(t)$, in which

$$\frac{1}{1-\gamma}(r + \lambda_{x+t}^S)\psi \geq \frac{1}{1-\gamma}\psi' + \delta\psi + \frac{\gamma}{1-\gamma}\psi^{\frac{1-\gamma}{\gamma}}, \quad (70)$$

with equality if

$$\frac{1}{1-\gamma}\psi(t) > \frac{1}{1-\gamma}\frac{\bar{a}_{x+t}^S}{(\bar{a}_{x+t}^O)^{1-\gamma}}. \quad (71)$$

The solution to this variational inequality is given by the bracketed term in equation (12) after one maximizes with respect to T , and we are done.

8.2 Appendix B: Impact of Objective vs. Subjective Health

In this appendix, we show that if the subjective force of mortality varies slightly from the objective force to the extent that $\bar{a}_x^S < 2\bar{a}_x^O$ for all x , then the optimal time of annuitization increases from the T given by the zero of the right-hand side of (16). In particular, if the individual is less healthy than normal ($\lambda_x^S > \lambda_x^O$ for all x), then $\bar{a}_x^S < \bar{a}_x^O$ for all x , from which it follows that the optimal time of annuitization is delayed. Also, if the individual is more healthy than normal but only to the extent that $\bar{a}_x^S < 2\bar{a}_x^O$ for all x , then the optimal time of annuitization is delayed.

Suppose $\bar{a}_{x+T}^S = \bar{a}_{x+T}^O + \varepsilon$ for some small ε , not necessarily positive. Then, equation (16) at the critical value T becomes

$$0 = \left[\frac{\gamma}{1-\gamma} \left(\frac{\bar{a}_{x+T}^O + \varepsilon}{\bar{a}_{x+T}^O} \right)^{-\frac{1-\gamma}{\gamma}} - \frac{1}{1-\gamma} + \frac{\bar{a}_{x+T}^O + \varepsilon}{\bar{a}_{x+T}^O} \right] + (\bar{a}_{x+T}^O + \varepsilon) [\delta - (r + \lambda_{x+T}^O)]. \quad (72)$$

We can simplify this equation to

$$\begin{aligned} 0 &= (\bar{a}_{x+T}^O + \varepsilon) [\delta - (r + \lambda_{x+T}^O)] - \frac{1}{2} \left(-\frac{1-\gamma}{\gamma} - 1 \right) \left(\frac{\varepsilon}{\bar{a}_{x+T}^O} \right)^2 \\ &\quad - \frac{1}{6} \left(-\frac{1-\gamma}{\gamma} - 1 \right) \left(-\frac{1-\gamma}{\gamma} - 2 \right) \left(\frac{\varepsilon}{\bar{a}_{x+T}^O} \right)^3 + \dots, \end{aligned} \quad (73)$$

if $\frac{\varepsilon}{\bar{a}_{x+T}^O}$ lies between -1 and 1 . Thus, by the mean value theorem, there exists ε^* between 0 and $\frac{\varepsilon}{\bar{a}_{x+T}^O}$ such that

$$0 = (\bar{a}_{x+T}^O + \varepsilon) [\delta - (r + \lambda_{x+T}^O)] + \frac{1}{2\gamma} (\varepsilon^*)^2,$$

or equivalently,

$$0 = [\delta - (r + \lambda_{x+T}^O)] + \frac{(\varepsilon^*)^2}{2\gamma \bar{a}_{x+T}^O (1 + \frac{\varepsilon}{\bar{a}_{x+T}^O})}. \quad (74)$$

The second term of the above equation is positive (and small) regardless of the sign of ε . Thus, T is determined by setting $[\delta - (r + \lambda_{x+T}^O)]$ equal to a negative number. It follows that, for λ_x^O increasing with respect to x , this value of T will be larger than the zero of (17).

Note that a sufficient condition for the above result is that $\frac{\varepsilon}{\bar{a}_{x+T}^O}$ lie between -1 and 1 . Without difficulty, one can show that this requirement is equivalent to $\bar{a}_{x+T}^S < 2\bar{a}_{x+T}^O$. For less healthy people ($\lambda_x^S > \lambda_x^O$ for all x), we have $\bar{a}_x^S < \bar{a}_x^O$ for all x , so $\bar{a}_{x+T}^S < 2\bar{a}_{x+T}^O$ holds automatically. Also,

there is some leeway in this inequality, so that even healthier people might have that $\bar{a}_{x+T}^S < 2\bar{a}_{x+T}^O$. Even when this inequality does not hold, we conjecture that we still have a delay in the time of annuitization beyond that given by the zero of the right-hand side of (17), as we see in one of the examples in Section 5.

8.3 Appendix C: Probability of Reduced Payments

The probability that consumption (as a percentage of initial wealth) at optimal time of annuitization is $p\%$ less than the consumption if one annuitizes immediately equals

$$\begin{aligned}
& \Pr \left(\frac{W_T^*}{\bar{a}_{x+T}^O} < (1 - 0.01p) \frac{w}{\bar{a}_x^O} \mid W_0 = w \right) \\
&= \Pr \left(w e^{\left(2\delta - r - \frac{(\mu-r)^2}{2\sigma^2\gamma^2}\right)T - \int_0^T k(s)ds + \frac{\mu-r}{\sigma\gamma} B_T} < (1 - 0.01p) w \frac{\bar{a}_{x+T}^O}{\bar{a}_x^O} \right) \\
&= \Pr \left(e^{\frac{\mu-r}{\sigma\gamma} B_T} < \frac{(1 - 0.01p) \bar{a}_{x+T}^O}{\bar{a}_x^O} e^{-\left(2\delta - r - \frac{(\mu-r)^2}{2\sigma^2\gamma^2}\right)T + \int_0^T k(s)ds} \right) \\
&= \Pr \left(B_T < \frac{\ln \left(\frac{(1-0.01p)\bar{a}_{x+T}^O}{\bar{a}_x^O} \right) - \left(2\delta - r - \frac{(\mu-r)^2}{2\sigma^2\gamma^2}\right)T + \int_0^T k(s)ds}{\frac{\mu-r}{\sigma\gamma}} \right) \\
&= \Phi \left(\frac{\ln \left(\frac{(1-0.01p)\bar{a}_{x+T}^O}{\bar{a}_x^O} \right) - \left(2\delta - r - \frac{(\mu-r)^2}{2\sigma^2\gamma^2}\right)T + \int_0^T k(s)ds}{\frac{\mu-r}{\sigma\gamma} \sqrt{T}} \right). \tag{75}
\end{aligned}$$

Here, Φ denotes the cumulative distribution function of the standard normal.

8.4 Appendix D: Variable and Fixed Annuity Benefits

In the body of the paper we assumed that the only immediate annuities available upon annuitization provided a fixed payout. In this appendix we examine a market in which annuity ‘wrappers’ are available on all asset classes with full asset allocation mobility. We investigate the optimal consumption, investment, and annuitization policies in this market. We introduce the symbol β to represent the proportion of wealth at the time of annuitization that is invested in the variable immediate annuity and $1 - \beta$ is the proportion invested in the fixed annuity. We assume that the mix between the variable and fixed annuities, namely β versus $1 - \beta$, stays fixed throughout the remaining life of the annuitant, which is a so-called *money mix* plan and has certain optimality features as shown by Charupat and Milevsky (2002). Again, we consider CRRA utility and provide

formulas for the power utility case. One can easily deal with the logarithmic case by letting λ approach 1 in the consumption, investment, and annuitization policies under power utility $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$, $\gamma = 1, \gamma \neq 1$.

To further capture the salient features of this product, we assume that the provider of the annuity has a nonzero insurance load on the fixed annuity such that the effective “return” on the fixed annuity is r' , with $r' \leq r$. Similarly, there is a nonzero insurance load on the variable annuity such that the drift on the variable annuity is μ' with $\mu' \leq \mu$ and $\mu' - r' \leq \mu - r$.

For the mixture of a variable and a fixed annuity, define the value function V by

$$V(w, t; T) = \sup_{\{c_s, \pi_s, \beta\}} E \left[\int_t^T e^{-r(s-t)} {}_{s-t}p_{x+t}^S \frac{1}{1-\gamma} c_s^{1-\gamma} ds \right. \quad (76)$$

$$\left. + \int_T^\infty e^{-r(s-t)} {}_{s-t}p_{x+t}^S \frac{1}{1-\gamma} \left(\frac{W_T}{\bar{a}_{x+T}^{O, r'}} e^{\beta \left(\mu' - r' - \frac{\beta \sigma^2}{2} \right) (s-T) + \beta \sigma (B_s - B_T)} \right)^{1-\gamma} ds \middle| W_t = w, \right]$$

in which the second superscript on $\bar{a}_{x+T}^{O, r'}$, namely r' , is the rate of discount used to calculate the actuarial present value of the annuity.

We can deal with the choice of β by noting that

$$E \left[W_T^{1-\gamma} e^{\beta(1-\gamma) \left(\mu' - r' - \frac{\beta \sigma^2}{2} \right) (s-T) + \beta(1-\gamma) \sigma (B_s - B_T)} \right] \quad (77)$$

$$= E \left[W_T^{1-\gamma} \right] e^{\beta(1-\gamma) \left(\mu' - r' - \frac{\beta \gamma \sigma^2}{2} \right) (s-T)}$$

Thus, the expectation is maximized if we maximize $\beta \left(\mu' - r' - \frac{\beta \gamma \sigma^2}{2} \right)$. It follows that the optimal value of β equals:

$$\beta^* = \frac{\mu' - r'}{\sigma^2} \quad (78)$$

Note that the optimal choice of β is independent of the optimal time T of annuitization. Naturally, if β^* is greater than one, the holdings are truncated by the seller of the annuity at 100% of the risky stock.

It follows that V solves the HJB equation

$$\left\{ \begin{array}{l} (r + \lambda_{x+t}^S) V = V_t + \max_{\pi} \left[\frac{1}{2} \sigma^2 p^2 V_{ww} + (\mu - r) \pi V_w \right] + r w V_w \\ \quad + \max_{c \geq 0} \left[-c V_w + \frac{1}{1-\gamma} c^{1-\gamma} \right], \\ V(w, T; T) = \frac{1}{1-\gamma} \left(\frac{w}{\bar{a}_{x+T}^{O, r'}} \right)^{1-\gamma} \bar{a}_{x+T}^{S, r - \frac{(1-\gamma)(\mu' - r')^2}{s \sigma^2 y}}. \end{array} \right. \quad (79)$$

Note that this is the same as the previous HJB equation, except that the boundary condition reflects the mixture of the variable and fixed annuities. The second superscript on the actuarial present value denotes the rate at which the annuity payments are discounted. It follows that V has the form as that given in equation (12), except that $\bar{a}_{x+T}^S = \bar{a}_{x+T}^{S,r}$ is replaced with:

$$\bar{a}_{x+T}^{S,r-\frac{(1-\gamma)(\mu'-r')^2}{s\sigma^2y}} \quad (80)$$

and $\bar{a}_{x+T}^O = \bar{a}_{x+T}^{O,r}$, is replaced with $\bar{a}_{x+T}^{O,r'}$. Also, note that the optimal investment policy is similar in form to the one given in the body of the paper for the fixed-only payout case. If there are no loads on the fixed and variable annuities, that is, if $r' = r$ and $\mu' = \mu$ then the proportion of wealth invested in the risky asset from before annuitization equals the proportion after annuitization; however, in this case, the optimal strategy of the individual is to annuitize immediately. This latter result follows from the work of Yaari (1965). For a CRRA investor (with no bequest motives) with no insurance loads on the annuities, the optimal mixture between risky and risk-free assets is invariant to whether the portfolio is annuitized or not.

The derivative of V with respect to T is proportional to

$$\begin{aligned} \frac{\delta V}{\delta T} \propto & \left[\frac{\gamma}{1-\gamma} \left(\frac{\bar{a}_{x+T}^{S,r-\frac{(1-\gamma)(\mu'-r')^2}{s\sigma^2y}}}{\bar{a}_{x+T}^{O,r'}} \right)^{\frac{1-\gamma}{\gamma}} - \frac{1}{1-\gamma} + \frac{\bar{a}_{x+T}^{S,r-\frac{(1-\gamma)(\mu'-r')^2}{s\sigma^2y}}}{\bar{a}_{x+T}^{O,r'}} \right] \\ & + \bar{a}_{x+T}^{S,r-\frac{(1-\gamma)(\mu'-r')^2}{s\sigma^2y}} [\delta - \delta' - \lambda_{x+T}^O], \end{aligned} \quad (81)$$

in which $\delta' = r' + \frac{(\mu'-r')^2}{2\sigma^2\gamma}$. We can use this equation to determine the optimal value of T in any given situation. In Table 6 we compare the optimal ages of annuitization and the imputed value of delaying when the individual can only buy a fixed annuity (compare with Table 1) and when the individual can buy a money mix of variable and fixed annuities.

Table 6 about here.

We assume that the financial market and mortality are as described in Section 5 except that for the variable annuity, the insurer has a 100-basis-point Mortality and Expense Risk Charge load on the return, so that the modified drift is $\mu' = 0.11$, and for the fixed annuity, the insurer has a 50-basis-point spread on the return, so that the modified rate of return is $r' = 0.055$. Assume that the individual has a CRRA of $\gamma = 2$, from which it follows that the individual will invest 75.0% in the risky stock before annuitization and 68.7% in the variable annuity after annuitization.

8.5 Appendix E: Escalating Annuity Benefit

Suppose that the individual can buy an escalating annuity. An escalating annuity is one for which the payments increase at a (constant) rate g . These are known as COLA (Cost Of Living Adjustment) annuities and are available from vendors that sell fixed annuities. These escalating annuities are popular as a hedge against (expected) inflation, since it is virtually impossible to obtain true inflation-linked annuities in the U.S. The actuarial present value of an escalating annuity can be written \bar{a}_x^{r-g} ; that is, the rate of discount r is reduced by the rate of increase of the payments g . As in the previous two sections, we consider CRRA utility and provide formulas for the power utility case. Thus, we can define the corresponding value function by

$$V(w, t; T) = \sup_{\{c_s, \pi_s, g\}} \left[E \int_t^T e^{-r(s-t)} {}_{s-t}p_{x+t}^S \frac{1}{1-\gamma} c_s^{1-\gamma} ds \right. \\ \left. + \int_T^\infty e^{-r(s-t)} {}_{s-t}p_{x+t}^S \frac{1}{1-\gamma} \left(\frac{W_T}{\bar{a}_{x+T}^{O, r-g}} e^{g(s-T)} \right)^{1-\gamma} ds \middle| W_t = w \right] \quad (82)$$

This expression is maximized with respect to g if

$$\frac{1}{1-\gamma} \frac{\int_0^\infty e^{-(r-(1-\gamma)g)s} ds}{\left[\int_0^\infty e^{-(r-g)s} {}_s p_{x+T}^O ds \right]^{1-\gamma}} \quad (83)$$

is maximized. The derivative of this expression with respect to g is proportional to

$$\frac{\int_0^\infty s e^{-(r-(1-\gamma)g)s} {}_s p_{x+T}^S ds}{\int_0^\infty e^{-(r-(1-\gamma)g)s} {}_s p_{x+T}^S ds} - \frac{\int_0^\infty s e^{-(r-g)s} {}_s p_{x+T}^O ds}{\int_0^\infty e^{-(r-g)s} {}_s p_{x+T}^O ds}. \quad (84)$$

Note that this is a difference of expectations of “ s ” with respect to two probability distributions. Also, note that if $\lambda_x^S = \lambda_x^O - c$ for all x and for some constant c , then the optimal value of g is $g^* = \frac{c}{\gamma}$. For example, if the individual is healthier to the extent that the subjective hazard rate is 0.01 less than the objective (pricing) hazard rate, then optimal rate of increase of the annuity payments (once the individual annuitizes his or her wealth) is $\frac{0.01}{\gamma}$. Note that in general, *healthier* people will want to buy escalating annuities with a *positive* g , while *sicker* people will want to buy escalating annuities with a *negative* g . This makes sense because healthier people anticipate living longer than normal, so they will be able to enjoy those larger annuity payments in the future. On the other hand, sicker people will not live as long, so they demand higher payments now.

Table 7 provides a numerical example for an individual who believes he or she is healthier than normal with $f = -0.5$; that is, the person has one-half the mortality rate of the average person. Suppose that the CRRA is $\gamma = 1.5$. We compare these numbers with those when the individual can

buy only a fixed annuity. It turns out that the optimal rate of escalation $g = 2\%$ (to two decimal places) for all ages and for both genders.

Table 7 about here.

Note that the individual is willing to annuitize earlier if there is a 2% escalating annuity available; however, there is still an advantage to wait.

Table 1

	$\gamma = 1$, Female (Male)			$\gamma = 2$, Female (Male)		
Age	Optimal Age	Value of delay	Prob. Lower Annuity	Optimal Age	Value of delay	Prob. Lower Annuity
60	84.5 (80.3)	44.0 (32.0)%	0.311 (0.353)	78.4 (73.0)	15.3 (8.9)%	0.268 (0.321)
65	84.5 (80.3)	33.4 (21.9)	0.346 (0.391)	78.4 (73.0)	10.3 (4.3)	0.310 (0.372)
70	84.5 (80.3)	22.7 (12.3)	0.385 (0.431)	78.4 (73.0)	5.2 (0.8)	0.362 (0.435)
75	84.5 (80.3)	12.3 (4.2)	0.429 (0.470)	78.4 (Now)	1.2 (0.0)	0.428 (N/a)
80	84.5 (80.3)	3.7 (0.02)	0.473 (0.500)	Now (Now)	neg. (neg.)	N/a (N/a)
85	Now (Now)	neg. (neg.)	N/a (N/a)	Now (Now)	neg. (neg.)	N/a (N/a)

All-or-Nothing Market: The value of the option to delay annuitization for males and females with a coefficient of relative risk aversion of $\gamma = 1$ and $\gamma = 2$. We assume the non-annuitized funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. The mortality is assumed to be Gompertz-Makeham fit to the IAM2000 table with projection Scale G. For example, a 70-year-old female with a coefficient of relative risk aversion of $\gamma = 2$, will effectively lose a 5.2% of her wealth if she chooses to annuitize immediately. The optimal time for her to annuitize is at age 78.4. The table also indicates the probability the annuity which is purchased at the optimal age will provide less income than the original annuity would have provided.

Table 2

	$\gamma = 1$, Female (Male)	$\gamma = 2$, Female (Male)
Age	Prob. of consuming at least 20% more	Prob. of consuming at least 20% more
60	0.644 (0.596)	0.631 (0.551)
65	0.602 (0.549)	0.565 (0.459)
70	0.552 (0.494)	0.474 (0.296)
75	0.493 (0.425)	0.316 (0.133)
80	0.414 (0.137)	N/a (N/a)
85	N/a (N/a)	N/a (N/a)

Assuming the individual self-annuitizes and defers the purchase of a life annuity to the optimal age, this table indicates the probability of consuming at least 20% more at the time of annuitization, compared to if one annuitizes immediately. Thus, for example, a female (male) at age 65 with a coefficient of relative risk aversion of $\gamma = 1$, has a 64.4% (59.6%) chance of creating a 20% larger annuity flow.

Table 3

f	Optimal age of annuitization	Value of Delay	Consumption rate Before Annuitization	Consumption rate After Annuitization
-1.0	78.28	13.79%	7.55%	13.38%
-0.8	74.58	10.54	7.95	11.79
-0.6	73.71	9.68	8.18	11.47
-0.4	73.29	9.23	8.37	11.33
-0.2	73.09	8.99	8.54	11.26
0.0	73.03	8.87	8.70	11.24
0.2	73.08	8.84	8.85	11.26
0.5	73.31	8.93	9.06	11.33
1.0	74.04	9.34	9.38	11.59
1.5	75.21	10.00	9.68	12.03
2.0	76.96	10.89	9.98	12.76
2.5	79.71	12.01	10.26	14.12
3.0	85.38	13.38	10.55	18.01

All-or-Nothing Market: The imputed value of delaying annuitization for a male, aged 60 with (CRRA) $\gamma = 2$. We assume the funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. The mortality is assumed to be Gompertz-Makeham fit to the IAM2000 table with projection Scale G. Thus, for example, if the individual's subjective hazard rate is 20% higher (i.e. less healthy) than the mortality table used by the insurance company to price annuities, the optimal annuitization point is at age 73.1, and the value of the option is 8.84% of the individual's wealth at age 60. While the 60-year-old male waits to annuitize, he consumes at the rate of 8.85% of assets, and once he purchases the fixed immediate annuity, his consumption rate - and standard of living - will increase to 11.26% of assets.

Table 4

Amount of Money Spent on Annuities for Various Levels of Wealth and Risk Aversion				
	$\gamma = 1.5$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
Wealth	$(z_0 = 3.273)$	$(z_0 = 2.354)$	$(z_0 = 1.837)$	$(z_0 = 1.506)$
\$1,000,000	\$727,620	\$792,020	\$831,852	\$858,901
\$500,000	\$331,384	\$371,251	\$395,909	\$412,653
\$250,000	\$133,266	\$160,866	\$177,937	\$189,529
\$100,000	\$14,395	\$34,635	\$47,154	\$55,655
\$50,000	\$0	\$0	\$3559	\$11,030

In an open market environment the table illustrates for various levels of relative risk aversion γ , the critical value of the ratio of liquid wealth to pre-existing annuity income $z_0 = w/A$ above which the individual will spend a lump-sum to increase her annuity income. We also include the amount the individual will spend on annuities for a given level of pre-existing annuity income $A = \$25,000$, namely $(w - z_0A)/(1 + (r + \lambda^O)z_0)$. We assume the following parameter value. The force of mortality $\lambda^S = \lambda^O = 0.04$, which implies a life expectancy of 25 years, the riskless rate of return is $r = 0.04$, the risky rate of return is $\mu = 0.08$, and the risky asset's volatility is $\sigma = 0.20$. Note that in contrast to the restricted all-or-nothing market, the individual immediately annuitizes a base level of income and then gradually annuitizes more as his wealth breaches higher levels.

Table 5

Amount of Money Spent on Annuities for Various Levels of Wealth and Risk Aversion				
	$\gamma = 1.5$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
Wealth	($z_0 = 3.273$)	($z_0 = 2.354$)	($z_0 = 1.837$)	($z_0 = 1.506$)
\$1,000,000	\$662,802	\$742,477	\$791,789	\$825,271
\$500,000	\$266,555	\$321,715	\$355,854	\$379,034
\$250,000	\$68,432	\$111,334	\$137,886	\$155,915
\$100,000	\$0	\$0	\$7,106	\$22,044
\$50,000	\$0	\$0	\$0	\$0

In this table we assume the same market and mortality parameters as in Table 4, but assume that the pre-existing annuity or pension income level is (doubled to) $A = \$50,000$. Once again we provide the critical value of the ratio of wealth to annuity income z_0 at which additional annuities are purchased. Intuitively, the greater the level of pre-existing annuity income the less liquid wealth must be annuitized to provide the optimal consumption stream.

Table 6

	Fixed Annuity, Female (Male)		Mixture of Variable (68.7%) and Fixed Annuities (31.3%), Female (Male)	
Age	Optimal age of annuitization	Value of delay	Optimal age of Annuitization	Value of delay
60	80.2 (75.2)	21.0 (13.4)%	70.8 (64.1)	3.4 (0.6)%
65	80.2 (75.2)	14.8 (7.5)	70.8 (Now)	1.3 (neg.)
70	80.2 (75.2)	8.5 (2.5)	70.8 (Now)	0.04 (neg.)
75	0.2 (75.2)	2.9 (0.003)	Now (Now)	neg. (neg.)

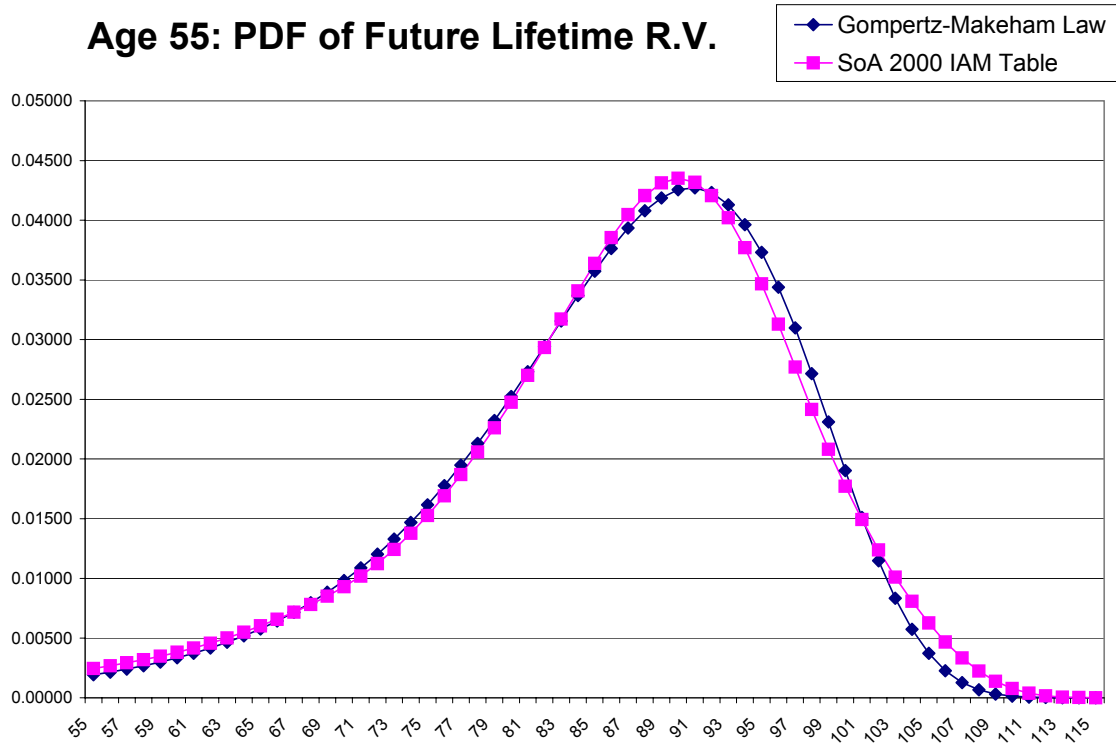
In an all-or-nothing annuitization environment – where both fixed and variable annuities are available with complete asset allocation flexibility – the table illustrates the imputed value of delaying annuitization for males and females with a Coefficient of Relative Risk Aversion of $\gamma = 2$. We assume that the non-annuitized funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. We introduce insurance loads on the variable and fixed annuities on the order of 100 basis points and 50 basis points respectively. The mortality is assumed to be Gompertz fit to the IAM2000 table with projection Scale G.

Table 7

	Fixed Annuity Female (Male)		2% Escalating Annuity Female (Male)	
Age	Optimal age of annuitization	Value of delay	Optimal age of annuitization	Value of delay
60	80.9 (76.1)	23.68 (15.59)%	78.5 (73.2)	17.41 (9.61)%
65	80.9 (76.1)	17.05 (9.24)	78.5 (73.2)	11.50 (4.88)
70	80.9 (76.1)	10.22 (3.57)	78.5 (73.2)	5.80 (0.96)
75	80.9 (76.1)	3.96 (0.15)	78.5 (Now)	1.29 (0.00)

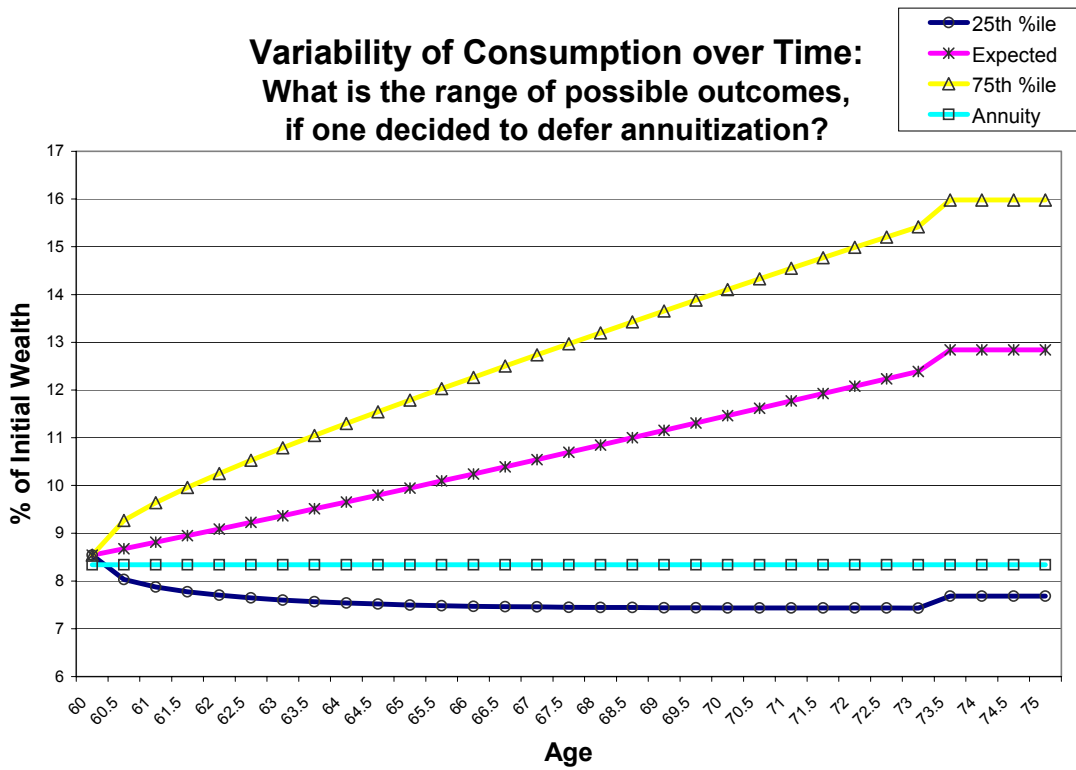
In an all-or-nothing annuitization environment, the table illustrates the value of delay for males and females with a coefficient of relative risk aversion (CRRA) of $\gamma = 1.5$. We assume the liquid funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. The mortality is Gompertz-Makeham fit to the IAM2000 table with projection Scale G, while the individual has subjective mortality beliefs equal to one-half of the objective mortality. Note that the availability of an increasing annuity – which better matches the desired consumption profile – accelerates the optimal age of annuitization and reduces the option value to wait.

Figure 1



The figure shows the probability density function (PDF) of the future-lifetime random variable under an analytic Gompertz-Makeham hazard rate that is fitted to the Individual Annuity Mortality Table 2000 with projection scale G. For males, the ‘best fitting’ parameters are $(m, b) = (88.18, 10.5)$ and for females they are $(92.63, 8.78)$.

Figure2



The figure shows the expected consumption of a 60-year-old male who believes he is 20% more healthy than average population rates; 8.34% is the rate of consumption if he annuitizes his wealth at age 60. We also display the 25th and 75th percentiles of the distribution of consumption between ages 60 and 75.

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