Optimal Annuity Purchasing

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Abstract:

We find the optimal annuity-purchasing scheme for an individual who seeks to maximize her expected utility of lifetime consumption and bequest. Milevsky and Young (2002) found the optimal time for an individual, who has no pre-existing annuities, to annuitize all her wealth. We now allow the individual to possess pre-existing annuities, to annuitize only a portion of her wealth at a given time, to buy annuities more than once -- even continuously, if that is optimal -- and to consume something other than the annuity income after annuitization.

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1. Introduction

We find the optimal annuity-purchasing scheme for an individual who seeks to maximize her expected utility of lifetime consumption and bequest. Milevsky and Young (2002) found the optimal time for an individual, who has no pre-existing annuities, to annuitize all her wealth. After annuitization, Milevsky and Young (2002) assumed that the individual consumes exactly the annuity income. By contrast, we allow the individual to possess pre-existing annuities, to annuitize only a portion of her wealth at a given time, to buy annuities more than once (even continuously, if that is optimal), and to consume something other than the annuity income after annuitization.

In Section 2, we present our model and argue that if the marginal utility of annuity income is larger than the adjusted marginal utility of wealth, then the individual will annuitize a lump sum. Thereafter, she will buy annuities at a continuous rate (possibly zero) in order to keep the marginal utility of annuity income no larger than the adjusted marginal utility of wealth. The marginal utility of wealth is adjusted by multiplying by the price of an annuity at a given age. Thus, the annuity-purchasing problem is qualitatively similar to the problem of optimal consumption and investment in the presence of proportional transaction costs. See Section 2 for details.

In Section 3, we examine the annuity-purchasing problem for the specific case of an individual with preferences that exhibit constant relative risk aversion (CRRA). We use the homogeneity property of the value function to reduce the problem's dimension by one. We study properties of the optimal consumption, investment, and annuity-purchasing policies. We show that if the wealth-to-income ratio is larger than a given number at time t, then the individual will purchase a lump sum annuity so that the wealth-to-income ratio equals that specific number. Conversely, if the wealth-to-income ratio is less than that number, then the individual will buy no annuity at that time. Section 4 concludes the paper.

2. Optimal Annuity Purchasing with General Preferences

In this section, we consider the optimal annuity-purchasing problem for an individual who seeks to maximize her expected utility of lifetime consumption and bequest. We assume that her preferences are rather general, and in Section 3, we specialize to the case for which preferences exhibit CRRA. We allow the individual to buy annuities in lump sums or continuously, whichever is optimal. Our results are similar to those of Dixit and Pindyck (1994, pp 359ff). They consider the problem of a firm's (irreversible) capacity expansion. For our individual, annuity purchases are also irreversible, and this leads to the similarity in results.

We assume that an individual can invest in a riskless asset whose price at time t, X_t, follows the process $dX_s = rX_s\, ds$, $X_t = x > 0$, for some fixed $r \geq 0$. Also, the individual can invest in a risky asset whose price at time s, S_s, follows geometric Brownian motion given by

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dB_s, \\ S_t = S > 0, \end{cases}$$

in which $\mu > r$, $\sigma > 0$, and B_s is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_s\}$ of the probability space $(\Omega, \mathcal{F}, \Pr)$. Let W_{s-} be the wealth at time s- of the individual (before purchasing annuities at that time), and let π_s be the amount that the decision maker invests in the risky asset at time s. The notation W_{s-} denotes the left-hand limit of wealth before the individual buys an annuity at that time; we allow annuity purchasing to occur in lump sums, if that is optimal. It follows that the amount invested in the riskless asset is $W_{s-} - \pi_s$. Also, the decision maker consumes at a rate of c_s at time s.

As for the actuarial assumptions, let ${}_t p_x^S$ denote the subjective conditional probability that an individual aged (x) believes he or she will survive to age ($x + t$). It is defined via the subjective hazard function, λ_{x+s}^S, by the formula ${}_t p_x^S = \exp\left(-\int_0^t \lambda_{x+s}^S \, ds \right)$. See Bowers et al. (1997) for further details on this notation. We have a similar formula for the objective conditional probability of survival, ${}_t p_x^O$, in terms of the objective hazard function, λ_{x+s}^O. The actuarial present value of a life annuity that pays \$1 per year continuously to (x) is written $\bar{a}_x$. It is defined by $\bar{a}_x = \int_0^\infty e^{-rt} \, {}_t p_x dt.$ If we use the subjective hazard rate to calculate the survival probabilities, then we write $\bar{a}_x^S$, while if we use the objective (pricing) hazard rate to calculate the survival probabilities, then we write $\bar{a}_x^O$. Just to clarify, by objective $\bar{a}_x^O$, we mean the actual market prices of the annuity, whereas $\bar{a}_x^S$ denotes what the market price "would be" were the insurance company to use the individual's subjective assessment of her mortality.

The individual has a non-negative annuity income *rate* at time s of A_{s-} before any annuity purchases at that time. We assume that she can purchase an annuity at the (unloaded) price of $\bar{a}_{x+s}^O$ per dollar of annuity income at time s, or equivalently, at age $x + s$. Thus, wealth follows the process

$$\begin{cases} dW_s = \left[rW_{s-} + (\mu - r)\pi_s + A_{s-} - c_s \right] ds + \sigma \pi_s dB_s - \bar{a}_{x+s}^O dA_s, \\ W_{t-} = w \geq 0. \end{cases} \tag{2.1}$$

Again, the negative sign for the subscript on wealth and annuities denotes the left-hand limit of those quantities before any (lump-sum) annuity purchases.

We assume that the decision maker seeks to maximize, over admissible $\{c_s, \pi_s, A_s\}$, her expected utility of lifetime consumption and bequest. Admissible $\{c_s, \pi_s, A_s\}$ are those that are measurable with respect to the information available at time s, namely $\mathcal{F}_s$, that restrict consumption and wealth to be non-negative, that restrict the annuity-income process to be non-negative and non-decreasing (i.e., annuity purchases are irreversible), and that result in (2.1) having a unique solution; see Karatzas and

Shreve (1998), for example. We also allow the individual to value expected utility via her subjective hazard rate (or force of mortality), while the annuity is priced by using the objective hazard rate.

Denote the random time of death of our individual by τ. Thus, the value function of the individual at time t, or age $x + t$, defined on $\bar{D} = \{(w, A, t) : w \geq 0, A \geq 0, t \geq 0\}$ is given by

$$\begin{aligned} U(w, A, t) &= \sup_{\{c_s, \pi_s, A_s\}} E\left[\int_t^\infty e^{-r(s-t)} \, {}_{s-t}p_{x+t}^S \, u_1(c_s) \, ds + e^{-r(\tau-t)} u_2(W_\tau) \, \middle| \, W_t = w, A_t = A \right] \\ &= \sup_{\{c_s, \pi_s, A_s\}} E\left[\int_t^\infty e^{-r(s-t)} \, {}_{s-t}p_{x+t}^S \left\{ u_1(c_s) + \lambda_{x+s}^S \, u_2(W_s) \right\} ds \, \middle| \, W_t = w, A_t = A \right], \end{aligned} \tag{2.2}$$

in which u_1 and u_2 are strictly increasing, concave utility functions of consumption and bequest, respectively. Note that we assume that the individual discounts consumption at the riskless rate r. If we were to model with a subjective discount rate of say ρ, then this is equivalent to using r as in (2.2) and adding $\rho - r$ to the subjective hazard rate. Thus, there is no effective loss of generality in setting the subjective discount rate equal to the riskless rate r.

In the following proposition, we present some properties of the value function.

Proposition 2.1: (*i*) *The value function U is jointly concave in w and A and strictly increasing with respect to both w and A.*

(*ii*) *The value function U is continuous on* $\bar{D}$.

Proof: The concavity of U follows from the concavity of u_1 and u_2 and from the linearity of (2.1) with respect to the controls. U is strictly increasing with respect to wealth because if the individual has wealth $w_1 > w_2$, then she can buy an annuity using $(w_1 - w_2)$ and consume an additional $\frac{w_1 - w_2}{\bar{a}_{x+t}^O}$ for the rest of her life. Therefore, she is better off than if she had only wealth w_2. Similarly, if the individual has annuity income $A_1 > A_2$, then she can consume an additional $(A_1 - A_2)$ for the rest of her life and be better off than if she had income A_2. The proof of the continuity of U follows from Shreve and Soner (1994). □

We continue with a formal discussion on the derivation of the associated Hamilton-Jacobi-Bellman equation. If $dA_s = \Delta A$ is a lump-sum purchase, then the HJB equation for the value function U is

$$(r+\lambda_{x+t}^S)U = U_t + (rw+A)U_w + \max_{c\geq 0}\left[u_1(c) - cU_w\right] + \max_{\pi}\left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww}\right]$$
$$+ \max_{0\leq \Delta A \leq \frac{w}{\overline{a}_{x+t}^O}}\left[U(w - \overline{a}_{x+t}^O\Delta A,\, A+\Delta A,\, t) - U(w, A, t)\right] + \lambda_{x+t}^S\, u_2(w). \tag{2.3}$$

See Björk (1998) for clear derivations of such HJB equations. The first-order necessary condition for ΔA gives us that ΔA solves the following equation:

$$U_A(w - \overline{a}_{x+t}^O\Delta A,\, A+\Delta A,\, t) = \overline{a}_{x+t}^O\; U_w(w - \overline{a}_{x+t}^O\Delta A,\, A+\Delta A,\, t). \tag{2.4}$$

Specifically, the lump-sum purchase is such that the marginal utility of annuity income equals the adjusted marginal utility of wealth. This is parallel to many results in economics. Indeed, the marginal utility of annuity income can be thought of as the marginal utility of the benefit, while the adjusted marginal utility of wealth can be thought of as the marginal utility of the cost. Thus, the lump-sum purchase forces the marginal utilities of benefit and cost to equal.

From Proposition 2.1, we know that the value function U is increasing and concave with respect to wealth w and annuity income A. Thus, if

$$U_A(w, A, t) > \overline{a}_{x+t}^O\; U_w(w, A, t), \tag{2.5}$$

then by decreasing wealth and increasing annuity income, we can achieve equality as in equation (2.4).

By following the arguments in Dixit and Pindyck (1994, pp 359ff) or in Zariphopoulou (1992), we discover that the optimal annuity-purchasing scheme is a type of "barrier control." Specifically, if inequality (2.5) holds at time t, then the individual will spend a lump-sum amount in order to reach equality as in (2.4). On the other hand, if at time t, we have

$$U_A(w, A, t) \leq \overline{a}_{x+t}^O\; U_w(w, A, t), \tag{2.6}$$

then the individual will buy annuities at a continuous rate (possibly zero) in order to maintain inequality (2.6).

Thus, the curve in wealth-annuity income space (w, A) at time t that is defined by the equality

$$U_A(w, A, t) = \bar{a}^O_{x+t}\, U_w(w, A, t) \tag{2.7}$$

can be thought of as a "barrier." If wealth and annuity income lie to the right of the barrier at time t, then the individual will immediately spend a lump sum of wealth to move diagonally to the barrier. The move is diagonal because as wealth decreases to purchase more annuities, annuity income increases. Thereafter, annuity income is either constant if wealth is low enough to keep to the left of the barrier, or annuity income responds continuously to infinitesimally small changes of wealth at the barrier.

In the region of no annuity purchasing, or *inaction*, namely where inequality (2.6) holds strictly, we have that the value function U satisfies the following HJB equation:

$$(r+\lambda^S_{x+t})U = U_t + (rw + A)U_w + \max_{c\ge 0}\left[u_1(c) - cU_w\right] + \max_{\pi}\left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww}\right] + \lambda^S_{x+t}\, u_2(w). \tag{2.8}$$

There exist verification theorems that tell us if the value function U is smooth and if $\tilde{U}$ is a smooth solution of the associated HJB equation, then under certain regularity conditions, $\tilde{U} = U$ (Fleming and Soner, 1993). However, in general, we can only assert that the value function is solves the HJB equation in the sense of viscosity solutions. Indeed, by following the arguments of Zariphopoulou (1992) or of Duffie and Zariphopoulou (1993), one can show that the value function is a constrained viscosity solution of the HJB equation under suitable regularity conditions. Specifically, we have the following proposition.

Proposition 2.2: *The value function U is a constrained viscosity solution on* $\bar{D}$ *of the Hamilton-Jacobi-Bellman equation*

$$\min\left[(r+\lambda^S_{x+t})U - U_t - (rw + A)U_w - \max_{c\ge 0}\left[u_1(c) - cU_w\right] - \max_{\pi}\left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww}\right] - \lambda^S_{x+t}\, u_2(w),\right.$$
$$\left.\bar{a}^O_{x+t} U_w - U_A\right] = 0.$$

□

In the next section, we analyze the HJB equation in (2.8) for the case of CRRA preferences.

3. Optimal Annuity Purchasing with CRRA Preferences

In this section, we specialize the results of Section 2 to the case for which the individual's preferences exhibit CRRA. Specifically, let

$$u_1(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \text{ and } u_2(c) = k \frac{1}{1-\gamma} c^{1-\gamma}, \ \gamma > 0, \gamma \neq 1, k \geq 0. \tag{3.1}$$

The parameter $k > 0$ weights the utility of bequest relative to the utility of consumption. Davis and Norman (1990) and Shreve and Soner (1994) show that for CRRA preferences in the problem of consumption and investment in the presence of transaction costs, the value function U is a solution of its HJB equation in the classical sense, not just in the viscosity sense. Generally, if the force of mortality is "eventually" large enough to make the value function well-defined, then this result holds for our problem, too.

For the utility functions in (3.1), it turns out that the value function U is homogeneous of degree $1 - \gamma$ with respect to wealth w and annuity income A. That is, $U(bw, bA, t) = b^{1-\gamma} U(w, A, t)$ for $b > 0$. Thus, if we define V by $V(z, t) = U(z, 1, t)$, then we can recover U from V by

$$U(w, A, t) = A^{1-\gamma} V(w/A, t), \text{ for } A > 0.$$

It follows that the HJB equation for U from Proposition 2.2 becomes the following equation for V:

$$\min\Bigg[(r + \lambda^S_{x+t})V - V_t - (rz+1)V_z - \max_{\hat{c} \geq 0}\left[\frac{\hat{c}^{1-\gamma}}{1-\gamma} - \hat{c}V_z\right] - \max_{\hat{\pi}}\left[(\mu - r)\hat{\pi}V_z + \frac{1}{2}\sigma^2\hat{\pi}^2 V_{zz}\right] - k\,\lambda^S_{x+t}\frac{z^{1-\gamma}}{1-\gamma}, \\ (z + \bar{a}^O_{x+t})V_z - (1-\gamma)V \Bigg] = 0, \tag{3.2}$$

in which $\hat{c} = \frac{c}{A}$, and $\hat{\pi} = \frac{\pi}{A}$. Davis and Norman (1990) and Shreve and Soner (1994) use the same transformation in the problem of consumption and investment in the presence of transaction costs. Also, Duffie et al. (1997) and Koo (1998) use this transformation to study optimal consumption and investment with stochastic income.

We next study properties of the optimal consumption and investment policies. Throughout we assume that the value function U is continuously twice differentiable and satisfies the HJB equation given in Proposition 2.2. Equivalently, we assume that the value function V is continuously twice differentiable and satisfies the HJB equation (3.2) for $z = w/A$. Thus, the optimal consumption and investment policies are given by the first-order necessary conditions.

We follow the work of Davis and Norman (1990) and Koo (1998) by expressing the value function U in an intuitively pleasing form.

Lemma 3.1: (*i*) *Define the function p in the region of inaction by*

$$p \equiv \frac{U_A(w, A, t)}{U_w(w, A, t)} \text{ for } w > 0, A > 0, t \geq 0.$$

Then, p is a function of the wealth-to-income ratio z = w/A and t. Define the function q in the region of inaction by

$$q \equiv \frac{(1-\gamma)U(w, A, t)}{(w + p(z, t)A)^{1-\gamma}} \text{ for } w > 0, A > 0, t \geq 0.$$

Then, q is also a function of z and t.

(*ii*) *The value function U in the region of inaction can be written as*

$$U(w, A, t) = \frac{q(z, t)}{1-\gamma}(w + p(z, t)A)^{1-\gamma} \text{ for } w > 0, A > 0, t \geq 0.$$

(*iii*) *The value function V in the region of inaction can be written as*

$$V(z, t) = \frac{q(z, t)}{1-\gamma}(z + p(z, t))^{1-\gamma} \text{ for } z > 0, t \geq 0.$$

Proof: The fact that p and q are functions of z and t follows from the homogeneity of U. The rest of the lemma is straightforward to show. □

It follows from (2.6), (2.7) and Lemma 3.1(ii) that $p(z,t) < \overline{a}^{O}_{x+t}$ if (w, A, t) is in the region of inaction, and $p(z,t) = \overline{a}^{O}_{x+t}$ if (w, A, t) is on the barrier, in which $z = w/A$.

Next, we give some properties of the functions p and q and use the first-order necessary conditions from (2.8) to derive the optimal consumption and investment policies in the region of inaction.

Proposition 3.2: (*i*) *In the region of inaction, the function p*(z, t) *is a non-decreasing function with respect to z, and the function q*(z, t) *satisfies*

$$\frac{q_z(z, t)}{q(z, t)} = -\frac{(1-\gamma)p_z(z, t)}{p(z, t)} \text{ for } z > 0, t \geq 0.$$

(*ii*) *In the region of inaction, the optimal consumption policy is given by*

$$c^*_t = q(Z^*_t, t)^{-\frac{1}{\gamma}}(W^*_t + p(Z^*_t, t)A^*_t),$$

in which W^*_t *and* A^*_t *are the optimally-controlled wealth and annuity income processes. Also,* $Z^*_t = W^*_t/A^*_t$.

(*iii*) *In the region of inaction, the optimal investment policy in the risky stock is given by*

$$\pi^*_t = \frac{\mu - r}{\sigma^2}\frac{1}{\gamma + p_z(Z^*_t, t)}(W^*_t + p(Z^*_t, t)A^*_t).$$

(*iv*) *Define the accounting total wealth by*

$$ATW(w, A, t) = w + A\,\bar{a}^O_{x+t}.$$

One can think of the accounting total wealth as the wealth required to have "liquid wealth" of w and an annuity income of A. Then, the ratio of consumption to accounting total wealth is increasing with respect to the wealth-to-income ratio.

Proof: For the sake of brevity, we omit the proof of this proposition. Please refer to Koo (1998) for details. □

From the fact that *p* is non-decreasing with respect to *z*, we have the following form for the barrier.

Proposition 3.3: *For each value of t* ≥ 0, *there exists a value of the wealth-to-income ratio* $z_0(t)$ *such that*

(*i*) *If* $z > z_0(t)$, *then the individual immediately buys an annuity so that*

$$\frac{w - \Delta A\,\bar{a}^O_{x+t}}{A + \Delta A} = z_0(t);$$

(*ii*) *If* $z < z_0(t)$, *then the individual buys no annuity; i.e., she is in the region of inaction.*

It follows that at each time point, the barrier is a ray emanating from the origin and lying in the first quadrant of (*w*, *A*) *space.*

Proof: If (w_1, A_1, t) is in the region of inaction, then for $z \le z_1 = w_1/A_1$, we have that

$$p(z, t) \le p(z_1, t) < \bar{a}^0_{x+t}.$$

Thus, any (w, A, t) such that $z < w/A$ is in the region of inaction. Let $z_0(t)$ solve the equation

$$p(z_0(t), t) = \bar{a}^O_{x+t}.$$

Then, $z_0(t)$ has the properties claimed in the proposition. □

Davis and Norman (1990) and Shreve and Soner (1994) find a similar result for the problem of optimal consumption and investment in the presence of proportional transaction costs.

We are now ready to give a more complete formulation of the value function U.

Proposition 3.4: *In the case of CRRA preferences for consumption and bequest* (3.1), *the value function U in* (2.2) *is given by*

$$U(w, A, t) = \begin{cases} \dfrac{q(z,t)}{1-\gamma}(w + p(z,t)A)^{1-\gamma} & \text{if } z = w/A < z_0(t), \\ \dfrac{q(z_0(t),t)}{1-\gamma}(w + \bar{a}^O_{x+t}A)^{1-\gamma} & \text{if } z = w/A \geq z_0(t), \end{cases}$$

in which p and q are given in Lemma 3.1, *and* $z_0(t)$ *is given in Proposition* 3.3.

Proof: The formulation of U for $z < z_0(t)$ follows from Lemma 3.1(ii) because we are in the region of inaction. If $z = z_0(t)$, then we are on the barrier, so by definition of $z_0(t)$, we have $p(z_0(t), t) = \bar{a}^O_{x+t}$. The form for U follows by continuity (Davis and Norman, 1990; Shreve and Soner, 1994). Finally, if $z > z_0(t)$, then the individual immediately buys an annuity so that

$$\frac{w - \Delta A\,\bar{a}^O_{x+t}}{A + \Delta A} = z_0(t),$$

from which it follows that

$$\begin{aligned} U(w, A, t) &= U(w - \Delta A\,\bar{a}^O_{x+t},\, A + \Delta A,\, t) \\ &= \frac{q(z_0(t),t)}{1-\gamma}\left[(w - \Delta A\,\bar{a}^O_{x+t}) + (A + \Delta A)\bar{a}^O_{x+t}\right]^{1-\gamma} \\ &= \frac{q(z_0(t),t)}{1-\gamma}(w + A\,\bar{a}^O_{x+t})^{1-\gamma}. \end{aligned}$$

□

4. Linearization of the HJB Equation with No Bequest Motive

In this section, we linearize the nonlinear partial differential equation for V in the region of inaction given by equation (3.2) with no bequest motive ($k = 0$). To do this, we consider the convex dual of V defined by

$$\tilde{V}(y,t) = \max_{z>0}\left[V(z,t) - zy\right]. \tag{4.1}$$

The critical value z^* solves the equation $0 = V_z(z,t) - y$; thus, $z^* = I(y,t)$, in which I is the inverse of V_z with respect to z. It follows that

$$\tilde{V}(y,t) = V[I(y,t),t] - yI(y,t). \tag{4.2}$$

Note that

$$\begin{aligned} \tilde{V}_y(y,t) &= V_z[I(y,t)]I_y(y,t) - I(y,t) - yI_y(y,t) \\ &= yI_y(y,t) - I(y,t) - yI_y(y,t) \\ &= -I(y,t). \end{aligned} \tag{4.3}$$

We can retrieve the function V from $\tilde{V}$ by the relationship

$$V(z,t) = \min_{y>0}\left[\tilde{V}(y,t) + zy\right]. \tag{4.4}$$

Indeed, the critical value y^* solves the equation $0 = \tilde{V}_y(y,t) + z = -I(y,t) + z$; thus, $y^* = V_z(z,t)$, and

$$\begin{aligned} \tilde{V}(y^*,t) + zy^* &= \tilde{V}[V_z(z,t),t] + zV_z(z,t) \\ &= V[I(V_z(z,t),t),t] - V_z(z,t)I(V_z(z,t),t) + zV_z(z,t) \\ &= V(z,t) - zV_z(z,t) + zV_z(z,t) \\ &= V(z,t), \end{aligned}$$

in which we use equation (4.2) for the second equality.

Next, note that

$$\tilde{V}_{yy}(y,t) = -I_y(y,t) = -1/V_{zz}[I(y,t),t], \tag{4.5}$$

and

$$\begin{aligned} \tilde{V}_t(y,t) &= V_z[I(y,t),t]I_t(y,t) + V_t[I(y,t),t] - yI_t(y,t) \\ &= yI_t(y,t) + V_t[I(y,t),t] - yI_t(y,t) \\ &= V_t[I(y,t),t]. \end{aligned} \tag{4.6}$$

In the partial differential equation for V with no bequest motive ($k = 0$), let $z = I(y,t)$ to obtain

$$\begin{aligned} (r + \lambda^S_{x+t})V[I(y,t),t] &= V_t[I(y,t),t] + (rI(y,t)+1)V_z[I(y,t),t] \\ &\quad + \frac{\gamma}{1-\gamma}(V_z[I(y,t),t])^{1-\frac{1}{\gamma}} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(V_z[I(y,t),t])^2}{V_{zz}[I(y,t),t]}. \end{aligned}$$

Rewrite this equation in terms of $\tilde{V}$ to get

$$(r+\lambda^{S}_{x+t})\left\{\tilde{V}(y,t)+yI(y,t)\right\}=\tilde{V}_t(y,t)+(-r\tilde{V}_y(y,t)+1)y+\frac{\gamma}{1-\gamma}y^{1-\frac{1}{\gamma}}-m\frac{y^2}{-1/\tilde{V}_{yy}(y,t)},$$

in which $m=\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2$, or equivalently,

$$-y-\frac{\gamma}{1-\gamma}y^{1-\frac{1}{\gamma}}=\tilde{V}_t-(r+\lambda^{S}_{x+t})\tilde{V}+\lambda^{S}_{x+t}y\tilde{V}_y+my^2\tilde{V}_{yy}, \tag{4.7}$$

with boundary conditions given implicitly by $V(0, t) = 1$ and $V(\bar{a}^{O}_{x+t}, t) = 0$. Note that (4.7) is a linear partial differential equation.

Next, consider the boundary conditions $p(z_0(t),t)=\bar{a}^{O}_{x+t}$ and from smooth pasting at the free-boundary, $p_z(z_0(t),t)=0$. We can write these in terms of V as

$$-(1-\gamma)V(z_0(t),t)+(z_0(t)+\bar{a}^{O}_{x+t})V_z(z_0(t),t)=0, \tag{4.8}$$

and

$$\gamma V_z(z_0(t),t)+(z_0(t)+\bar{a}^{O}_{x+t})V_{zz}(z_0(t),t)=0. \tag{4.9}$$

We also have a boundary condition at $z = 0$ because at that point, the individual has no wealth to invest in the risky asset; thus, $V_{zz}(0, t) = -\infty$ for all $t \geq 0$. Because $V_z > 0$ is strictly decreasing with respect to z, we have $y_a(t) > y_0(t) \geq 0$ for all $t \geq 0$, in which $y_a(t)$ and $y_0(t)$ are defined by

$$y_a(t)=V_z(0,t), \tag{4.10}$$

and

$$y_0(t)=V_z(z_0(t),t). \tag{4.11}$$

Thus, in terms of $\tilde{V}$, the boundary conditions become

$$\tilde{V}_{yy}(y_a(t),t)=0, \text{ for } \tilde{V}_y(y_a(t),t)=0, \tag{4.12}$$

and

$$(1-\gamma)\tilde{V}(y_0(t),t)+\gamma y_0(t)\tilde{V}_y(y_0(t),t)=\bar{a}^{O}_{x+t}y_0(t), \tag{4.13a}$$

$$\text{for } \tilde{V}_y(y_0(t),t)+\gamma y_0(t)\tilde{V}_{yy}(y_0(t),t)=\bar{a}^{O}_{x+t}. \tag{4.13b}$$

To solve the second-order partial differential equation (4.7) with free boundaries determined by (4.12) and (4.13), we propose the following algorithm. First, suppose we have estimates of the functions y_0 and y_b. Use these estimates to solve the partial differential equation (4.7) with normal conditions at y_0 and y_b given by the second equations of (4.12) and (4.13). Re-estimate y_0 and y_b by using the first equations of (4.12) and (4.13), and repeat the process until it converges. Implementing this algorithm is the subject of future research. In the next section, we solve the system in the case for which the forces of mortality are constant.

5. Constant Forces of Mortality

5.1 Solution of the Boundary-Value Problem

If we assume that the forces of mortality are constant, that is, $\lambda^S_{x+t} \equiv \lambda^S$ and $\lambda^O_{x+t} \equiv \lambda^O$ for all $t \geq 0$, then we can obtain an "implicit" analytical solution of the value function V via the boundary-value problem given by (4.7), (4.12), and (4.13). In this case, V, $\tilde{V}$, y_a, and y_0 are independent of time, so (4.7) becomes the ordinary differential equation:

$$-y - \frac{\gamma}{1-\gamma} y^{1-\frac{1}{\gamma}} = -(r+\lambda^S)\tilde{V}(y) + \lambda^S y \tilde{V}'(y) + m y^2 \tilde{V}''(y), \tag{5.1}$$

with boundary conditions

$$\tilde{V}''(y_a) = 0, \text{ for } \tilde{V}'(y_a) = 0, \tag{5.2}$$

and

$$(1-\gamma)\tilde{V}(y_0) + \gamma y_0 \tilde{V}'(y_0) = \frac{y_0}{r+\lambda^O}, \tag{5.3a}$$

$$\text{for } \tilde{V}'(y_0) + \gamma y_0 \tilde{V}''(y_0) = \frac{1}{r+\lambda^O}. \tag{5.3b}$$

The general solution of (5.1) is

$$\tilde{V}(y) = D_1 y^{B_1} + D_2 y^{B_2} + \frac{y}{r} + C_2 y^{1-\frac{1}{\gamma}}, \tag{5.4}$$

with D_1 and D_2 constants to be determined by the boundary conditions, with C_2 given by

$$C_2 = r + \frac{\lambda^S}{\gamma} - m\frac{1-\gamma}{\gamma^2}, \tag{5.5}$$

and with B_1 and B_2 given by

$$B_1 = \frac{1}{2m}\left[(m-\lambda^S)+\sqrt{(m-\lambda^S)^2+4m(r+\lambda^S)}\right] > 1, \tag{5.6}$$

and

$$B_2 = \frac{1}{2m}\left[(m-\lambda^S)-\sqrt{(m-\lambda^S)^2+4m(r+\lambda^S)}\right] < 0. \tag{5.7}$$

The boundary conditions at y_0 give us

$$D_1\left\{1+\gamma(B_1-1)\right\}y_0^{B_1}+D_2\left\{1+\gamma(B_2-1)\right\}y_0^{B_2}+\frac{y_0}{r}=\frac{y_0}{r+\lambda^O}, \tag{5.8}$$

and

$$D_1B_1\left\{1+\gamma(B_1-1)\right\}y_0^{B_1}+D_2B_2\left\{1+\gamma(B_2-1)\right\}y_0^{B_2}+\frac{y_0}{r}=\frac{y_0}{r+\lambda^O}. \tag{5.9}$$

Solve equations (5.8) and (5.9) to get D_1 and D_2 in terms of y_0:

$$D_1 = -\frac{\lambda^O}{r(r+\lambda^O)}\frac{1-B_2}{B_1-B_2}\frac{y_0^{1-B_1}}{1+\gamma(B_1-1)}, \tag{5.10}$$

and

$$D_2 = -\frac{\lambda^O}{r(r+\lambda^O)}\frac{B_1-1}{B_1-B_2}\frac{y_0^{1-B_2}}{1+\gamma(B_2-1)}. \tag{5.11}$$

Next, substitute for D_1 and D_2 in $\tilde{V}'(y_a)+\gamma y_a\tilde{V}''(y_a)=0$ from (5.2), specifically $D_1B_1\left\{1+\gamma(B_1-1)\right\}y_a^{B_1-1}+D_2B_2\left\{1+\gamma(B_2-1)\right\}y_a^{B_2-1}+\frac{1}{r}=0,$ to get

$$\frac{\lambda^O}{r+\lambda^O}\frac{B_1(1-B_2)}{B_1-B_2}\left(\frac{y_a}{y_0}\right)^{B_1-1}+\frac{\lambda^O}{r+\lambda^O}\frac{B_2(B_1-1)}{B_1-B_2}\left(\frac{y_a}{y_0}\right)^{B_2-1}=1. \tag{5.12}$$

(5.12) gives us an equation for the ratio $y_a/y_0>1$. To check that (5.12) has a unique solution greater than 1, note that the left-hand side (1) equals $\lambda^O/(r+\lambda^O)<1$ when we set $y_a/y_0=1$, (2) goes to infinity as y_a/y_0 goes to infinity, and (3) is strictly increasing with respect to y_a/y_0.

Next, substitute for D_1 and D_2 in $\tilde{V}'(y_a)=0$ from (5.2), specifically $D_1B_1y_a^{B_1-1}+D_2B_2y_a^{B_2-1}+\frac{1}{r}+C_2\left(1-\frac{1}{\gamma}\right)y_a^{-\frac{1}{\gamma}}=0,$ to get

$$-\frac{\lambda^O}{r(r+\lambda^O)}\frac{B_1(1-B_2)}{B_1-B_2}\frac{\left(y_a/y_0\right)^{B_1-1}}{1+\gamma(B_1-1)}-\frac{\lambda^O}{r(r+\lambda^O)}\frac{B_2(B_1-1)}{B_1-B_2}\frac{\left(y_a/y_0\right)^{B_2-1}}{1+\gamma(B_2-1)}+\frac{1}{r}+C_2\left(1-\frac{1}{\gamma}\right)y_a^{-\frac{1}{\gamma}}=0. \tag{5.13}$$

Substitute for y_a/y_0 in equation (5.13), and solve for y_a. Finally, we can get y_0 from

$$y_0=\frac{y_a}{y_a/y_0}, \tag{5.14}$$

and D_1 and D_2 from equations (5.10) and (5.11), respectively.

Once we have the solution for $\tilde{V}$, we can recover V from

$$\begin{aligned} V(z) &= \max_{y>0}\left[\tilde{V}(y)+zy\right] \\ &= \max_{y>0}\left[D_1 y^{B_1}+D_2 y^{B_2}+\frac{y}{r}+C_2 y^{1-\frac{1}{\gamma}}+zy\right], \end{aligned} \tag{5.15}$$

in which the critical value y^* solves

$$D_1 B_1 y^{B_1-1}+D_2 B_2 y^{B_2-1}+\frac{1}{r}+C_2\left(1-\frac{1}{\gamma}\right)y^{-\frac{1}{\gamma}}+z=0. \tag{5.16}$$

Thus, for a given value of $z = w/A$, solve (5.16) for y and plug that value of y into (5.15) to get $U(w, A) = V(z)$. Perhaps more importantly, we are interested in the critical value z_0 above which an individual spends a lump-sum to purchase more annuity income.

5.2 *Numerical Examples*

In this section, we present numerical examples to demonstrate the results of Section 5.1.

Example 5.1: Suppose we have the following values of the parameters:

- $\lambda^S = \lambda^O =$ 0.04; the force of mortality is constant such that the expected future lifetime is 25 years.
- r = 0.04; the riskless rate of return is 4%.
- μ = 0.08; the risky rate of return is 8%.
- σ = 0.20; the risky asset's volatility is 20%.

In Table 1, for various values of γ, we give the critical value of the ratio of wealth to annuity income $z_0 = w/A$ above which individual will spend a lump-sum of wealth to increase her annuity income. We also include the amount that the individual will spend on annuities for a given annuity income of A = \$25,000: $(w-z_0A)/(1+(r+\lambda^O)z_0)$.

Table 1. Amount of Money Spent on Annuities for Various Levels of Wealth and Risk Aversion				
Wealth	$\gamma = 1.5$ ($z_0 = 3.273$)	$\gamma = 2.0$ ($z_0 = 2.354$)	$\gamma = 2.5$ ($z_0 = 1.837$)	$\gamma = 3.0$ ($z_0 = 1.506$)
\$1,000,000	\$727,620	\$792,020	\$831,852	\$858,901
\$500,000	\$331,384	\$371,251	\$395,909	\$412,653
\$250,000	\$133,266	\$160,866	\$177,937	\$189,529
\$100,000	\$14,395	\$34,635	\$47,154	\$55,655
\$50,000	\$0	\$0	\$3559	\$11,030

Notice from Table 1 that the amount spent on annuities increases, for a given level of wealth, as the individual becomes more risk averse, an intuitively pleasing result. Also, for a given level of risk aversion, the amount spent on annuities decreases as wealth decreases.

Example 5.2: In this example, we examine how the critical ratio z_0 changes as the parameters change. We take given the values of the parameters as in Example 5.1 with $\gamma = 2.0$ and determine z_0 by varying one parameter at a time and keeping all the parameters fixed. In a sense, we did this for the coefficient of relative risk aversion γ in Example 5.1. In Tables 2 through 7 we calculate z_0 for changes in λ^S, $\lambda^S = \lambda^O$, λ^O, r, μ, and σ, respectively.

From Table 2, we see that z_0 increases monotonically with respect to λ^S because as the individual becomes healthier relative to the objective mortality used in pricing the annuities, annuities become less attractive. Therefore, the individual will not annuitize as much of her wealth.

Table 2. Critical Ratio z_0 as a Function of λ^S	
λ^S	z_0
0.00	1.637
0.01	1.768
0.02	1.925
0.03	2.117
0.04	2.354
0.05	2.644
0.06	2.999
0.07	3.423
0.08	3.919
0.09	4.489
0.10	5.133
0.12	6.639
0.15	9.470

0.20	15.95
0.25	25.25
0.30	38.51
0.40	84.38
0.50	178.7
1.00	7,779

From Table 3, we see that the critical ratio z_0 monotonically decreases with respect to $\lambda^S = \lambda^O$ because annuities become less attractive to the investor as mortality rates increase.

Table 3. Critical Ratio z_0 as a Function of $\lambda^S = \lambda^O$

$\lambda^S = \lambda^O$	z_0
0.01	11.83
0.02	5.664
0.03	3.451
0.04	2.354
0.05	1.719
0.06	1.315
0.07	1.041
0.08	0.845
0.09	0.701
0.10	0.591
0.12	0.437
0.15	0.299
0.20	0.181
0.25	0.122
0.30	0.087
0.40	0.051
0.50	0.034
1.00	0.009

From Table 4, we see that the critical ratio z_0 decreases with respect to λ^O for the same reason as in Table 3. However, the decrease is more marked here because the individual's subjective mortality is constant.

Table 4. Critical Ratio z_0 as a Function of λ^O

λ^O	z_0
0.01	20.93
0.02	7.709
0.03	3.930
0.04	2.354
0.05	1.561
0.06	1.111

0.07	0.832
0.08	0.647
0.09	0.519
0.10	0.425
0.12	0.301
0.15	0.197
0.20	0.113
0.25	0.074
0.30	0.052
0.40	0.030
0.50	0.019
1.00	0.005

From Table 5, we see that the critical ratio z_0 decreases as a function of $r < \mu$ because as r increases, annuities become more attractive relative to the risky asset.

Table 5. Critical Ratio z_0 as a Function of r

r	z_0
0.01	13.65
0.02	7.623
0.03	4.297
0.04	2.354
0.05	1.188
0.06	0.497
0.07	0.124
0.075	0.032
0.078	0.005
0.079	0.001
0.08	0.000

From Table 6, we see that the critical ratio z_0 increases as a function of μ because as μ increases, the risky asset becomes more attractive relative to annuities.

Table 6. Critical Ratio z_0 as a Function of μ

μ	z_0
0.01	0.000
0.02	0.000
0.03	0.000
0.04	0.000
0.05	0.174
0.06	0.638
0.07	1.361
0.08	2.354
0.09	3.652
0.10	5.318

0.11	7.439
0.12	10.14
0.13	13.61
0.14	18.09
0.15	23.93

From Table 7, we see that the critical ratio z_0 decreases as a function of σ because as σ increases, the risky asset becomes more volatile and thereby the risky asset becomes less attractive relative to annuities.

Table 7. Critical Ratio z_0 as a Function of σ

σ	z_0
0.03	7,909
0.04	404.2
0.05	102.4
0.06	46.17
0.07	26.97
0.08	18.09
0.09	13.18
0.10	10.14
0.12	6.674
0.15	4.163
0.20	2.354
0.25	1.537
0.30	1.091
0.40	0.638
0.50	0.420
1.00	0.114

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