The Term Structure of Mortality-Contingent Claims: Some Canadian Evidence

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Version: August 8, 2001

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ABSTRACT

From a financial economic perspective, a single premium immediate life annuity (SPIA) can be viewed as defaultable coupon bearing bond that ceases payment upon death of the annuitant. The interest rate and hazard process are independent, the recovery rate is zero, and eventual default is certain.

Motivated by this analogy, this paper models the evolution of the term structure of mortality-contingent claims using a two factor Longstaff-Schwartz (1992) specification. We calibrate this model to a unique Canadian database containing a cross sectional time series of SPIA quotes from 1985 - 2001.

Our methodology differs from other Money's Worth studies, by letting the data imply the insurance loadings as well as mortality parameters. We ask the market to determine mortality expectations, which we then track over a 16-year time period.

Aside from the methodological contribution, our main results are that (i) market implied mortality (hazard) rates have been declining over the last 16 years, and (ii) the profitability (mark-up, or loading) for life annuities is on the order of 2% - 3% relative to the risk-free return in Canada, and (iii) the implied hazard rates fluctuate substantially, which we interpret as evidence that immediate annuity quotes do not instantaneously adjust to changes in the yield curve.

These results have direct implications for the valuation of long-term options on mortality-contingent claims, which are embedded in many insurance savings products.

JEL Classification: D91, G11

Keywords: Life Annuities, Yield Curve.

1 Introduction and Motivation

At the age of retirement, most individuals must decide if and when to acquire life annuities, or additional pension income. In exchange for an initial (non-reversible) lump sum, a life annuity guarantees periodic income for the rest of ones life. This is why annuities are often referred to as longevity insurance, since they allow one to hedge the risk of living too long. Fixed immediate annuities can be purchased in the competitive open market, at various ages and with various guarantees. In Canada, between \$650 Million and \$1 Billion (CDN) of these are sold annually. Source: Aronoff (2000).

To a financial economist, a life annuity can be viewed as a fixed-income instrument, or coupon bearing bond, that terminates, or defaults, upon death of the annuitant. There is no recovery upon default, and the arrival of this event is stochastic. With this analogy in mind, one can employ a variety of techniques and models that have been proposed for analyzing defaultable bonds. And, since the default process and the underlying interest rate process must be independent, the annuity market provides us with a 'clean' environment for testing alternative pricing models for defaultable bonds.

Indeed, the annuity market has garnered much economic attention in recent years. The interest has been primarily driven by proposals to reform Social Security, but also a result of the various annuitization guarantees that are sold by insurance companies in their retirement savings products[1]. Both industry and government are struggling with the same question of: "What will annuities cost in the future?" Our research is a step in that direction by examining the evolution of these prices in Canada during the last 16 years.

To this end, we have compiled a unique and comprehensive database that tracks the evolution of single premium immediate annuity (SPIA) prices in Canada across time and space (age). Our prices have been updated monthly (from 1985 - 2000) and weekly (from 2000 - 2001), and include all insurance companies in Canada that sell annuities. The dataset we are using consists of the annuity payments for both male and females of different ages, from age 55 to age 80 with 5-year increments, as well as different number of years of guarantee, from 0 years to 20 years with 5-year increments. Together with the set of zero-coupon Canadian bond yields, we have a rich set of information. In parallel with the finance literature, we refer to this collection as the term structure of mortality-contingent claims.

Assuming a No Arbitrage framework, the price of an SPIA is simply the discounted (risk-neutral) expectation of its future payoff. Therefore, pursuant to the analogy of defaultable bonds, there exists an implied hazard-plus-interest rate process off-which the annuity is priced. We locate this implied rate by assuming it obeys a particular functional form known

[1]See Milevsky and Promislow (2001), or Milevsky and Posner (2001) for a discusion of these guarantees within the context of a No Aribtrage model.

$_{(t-55)}p_{55}$	**1971** IAM		**1983** IAM		**1996** IAM	
t	F.	M.	F.	M.	F.	M.
55	1.00	1.00	1.00	1.00	1.00	1.00
60	.976	.952	.982	.966	.985	.974
65	.938	.886	.956	.919	.962	.937
70	.889	.799	.914	.848	.926	.880
75	.812	.682	.849	.742	.899	.791
80	.689	.530	.745	.596	.775	.663
85	.504	.353	.586	.415	.628	.496
90	.281	.181	.379	.234	.427	.313
95	.103	.056	.181	.100	.221	.154
100	.026	.007	.059	.028	.082	.055

Table 1: Annuitant (Period) Mortality Tables: Evolution Over Time

as the Gompertz specification. However, our estimation procedure can be applied to any functional form for mortality, provided we have enough annuity quotes to 'back out' unique mortality function parameters.

Our methodology differs from other Money's Worth studies in the economics literature, which was originally pioneered by Friedman and Warshawsky (1990) and recently updated by Mitchell, Poterba, Warshawsky and Brown (1999), in the United States, and Kim and Sharp (1999) as well as Milevsky (1998) in Canada. Those papers use a particular mortality table to 'back out' the implied insurance loadings from annuities. Their results are quite sensitive to the projection factors and specific mortality table.

In this paper, we let the actual data imply the insurance loadings as well as mortality parameters as opposed to exogenously imposing a particular actuarial table. This allows the market to "tell us" mortality expectations, which we then track over a 16-year time period. This is similar to work by Mullin and Philipson (1997) with life insurance contracts. Although, in this paper, we use the universe of annuity contracts in Canada over a long period of time, which allows for cross-sectional and time varying effects.

Aside from the methodological contribution, our results confirm that market-implied mortality (hazard) rates have been declining over the last 16 years. This is consistent with improvements in mortality that have appeared in tables compiled by the Society of Actuaries and Statistics Canada. Table #1 examines the change in mortality rates during the last 30 years, as documented by a particular mortality table known as the Individual Annuity Mortality Table.

Furthermore, it appears that the profitability – also known as mark-up, or loading – for life annuities in Canada is on the order of 2% - 3%, when compared to the risk-free rate. This loading is lower than the rates derived by Mitchell, Poterba, Warshawsky and Brown (1999) for the United States, but is consistent with the numbers derived by Kim and Sharp (1999) in Canada. We further confirm a declining trend in implied hazard rates and survival probabilities during the last 16 year. Finally, we interpret the volatility in implied hazard rates as possible evidence that annuity vendors do not instantaneously adjust their prices to changes in the risk-free yield curve. We argue that implied hazard rates fluctuate 'too much' to justify perfect correlation between the two curves.

The remainder of this paper is organized as follows. Section 2 reviews some basic actuarial theory, Section 3 describes the Longstaff-Schwartz (1992) as it applies to mortality contingent claims. Section 4 describes the unique database we have compiled, while section 5 uses the General Method of Moments to estimate the parameters of the model and Section 6 concludes the paper.

2 Review of Some Actuarial Theory

This section reviews some introductory material on the pricing of simple mortality-contingent claims. Throughout the paper, we assume that the time-at-death random variable, denoted by T, can be expressed in a continuous-time manner. For an individual currently aged x, the probability of death prior to time $t \geq 0$ (i.e., prior to age $x + t$) is modeled as:

$$\Pr(T \leq t \mid x) := 1 - ({}_tp_x) := 1 - e^{-\int_x^{x+t} h_s ds} \tag{1}$$

where h_s is the force-of-mortality, or hazard rate, and ${}_tp_x$ is the conditional probability that an individual aged x will survive for t years. The functional $h_s ds$ can heuristically be described as the *instantaneous* probability of death, applicable at time s. The conditional probability that an x-year-old individual will survive for t-years, is clearly greater than the probability of surviving for t-plus one year. Also, by definition, ${}_0p_x = 1$ and ${}_\infty p_x = 0$.

Equation (1) should be interpreted as a proper cumulative distribution function (CDF), and denoted by $F_x(t)$, provided that the hazard rate integrates to infinity. We let $f_x(t)$ denote the probability density function (PDF) of the time-at-death random variable, for an individual aged x. It follows, therefore, from (1) that:

$$\int_0^\infty ({}_tp_x) h_t dt = 1. \tag{2}$$

A simple application of the chain rule retrieves the convenient relationship:

$$h_{x+t} = \frac{f_x(t)}{1 - F_x(t)}. \tag{3}$$

A realistic continuous-time hazard rate – and the one we shall use in this paper – is the Gompertz mortality law. This is the same functional form used by Mullin and Philipson (1997) in their analysis of insurance prices. The exact specification is:

$$h_{x+t} = h_x e^{gt}, \tag{4}$$

Accordingly, as per equation (1), the conditional probability of survival, is:

$${}_tp_x = e^{-\int_x^{x+t} h_s ds} = e^{-\int_0^t h_x e^{gs} ds} = e^{-h_x\left(\frac{e^{gt}-1}{g}\right)} \tag{5}$$

We refer the interested reader to Carriere (1994) for a discussion of the 'goodness of fit' of the Gompertz distribution at various ages[2].

3 The Life-Annuity Pricing Model.

3.1 The interest rate component.

There are essentially two approaches to the modeling of the term structure of interest rate, i.e. the equilibrium approach pioneered by Cox, Ingersoll, and Ross (1985) and the arbitrage approach used in Ho and Lee (1986) and Heath, Jarrow and Morton (1988). The equilibrium approach has several clear advantages over the arbitrage approach. For example, under the equilibrium approach both the term structure of interest rate and its dynamics, as well as the prices of contingent claims, are endogenously determined. In contrast, the arbitrage approach provides no guidance as to the form of the factor risk premiums. Consequently, the arbitrary choice of the functional form can lead to internal inconsistency or arbitrage opportunities (see e.g. CIR, 1985b, Backus, Foresi and Zin, 1998). In this paper we use the Longstaff-Schwartz (1992) two-factor process, which is developed under a general equilibrium framework, to model the dynamics of Canadian interest rate term structure. The model is chosen among other two-factor interest rate candidate models, such as the Brennan and Schwartz (1979), Schaefer and Schwartz (1984), or Cox, Ingersoll and Ross (1985) for the following reasons. First of all, the two-factor model is motivated by the factor analysis of the historical Canadian term structure data and the parsimonious of the model structure. The factor analysis of the Canadian historical weekly interest rate term structure (from January, 1985 to May, 2001) provides similar findings for the U.S. interest rate term structure (see Litterman and Scheikman, 1991). Our results also identify three major factors along the Canadian yield curve, namely a level factor, a steepness factor and a curvature factor. The

[2]Some argue that at higher ages the hazard rate, or force of mortality, levels off, or perhaps even become flat. Likewise, at young ages the hazard rate increases and then declines. But for the annuitant population, from 50 to 80, we believe that Gompertz is reasonably accurate.

proportions of total explained variation accounted by each of the above three factors are respectively 70.83%, 17.39% and 6.15%. The findings suggest that the first two factors account for nearly 90% of the variation of the term structure dynamics, with the first factor clearly being the short-term interest rate. Second, evidence in Dybvig (1989) suggests that short term interest rate changes and interest rate volatility are the two most important factors in explaining movements in the term structure, which is consistent with the Longstaff-Schwartz (1992) two-factor model. Furthermore, the Longstaff-Schwartz (1992) two-factor model has been shown to provide a better fit to the cross section of monthly yield changes in the U.S. (see e.g. the empirical study by Baums, 1998).

According to the Longstaff-Schwartz (1992) model, the two 'factors' are the instantaneous short rate r, and instantaneous variance V. The diffusion process for these variables under the risk-neutral measure is:

$$dx = (\gamma - \delta x)dt + \sqrt{x}dW_1 \tag{6}$$

$$dy = (\eta - \nu y)dt + \sqrt{y}dW_2 \tag{7}$$

$$r = \alpha x + \beta y \tag{8}$$

$$V = \alpha^2 x + \beta^2 y \tag{9}$$

This model contains six free parameters $\{\alpha, \beta, \gamma, \delta, \eta, \nu\}$, that are all positive. In the event that $\alpha = \delta = \gamma = 0$, one can easily see that the model 'loses' one of the factors – since x becomes redundant – and it collapses to the standard Cox-Ingersoll-Ross (1985) specification. Also, we note that the real world diffusion for the correlated pair $\{r, V\}$ would have the parameter ν augmented by the market price of risk λ.

The long-run stationary distribution of r has a mean of $E[r] = \alpha\gamma/\delta + \beta\eta/\nu$ and a variance of $Var[r] = \alpha^2\gamma/2\delta^2 + \beta^2\eta/2\nu^2$. Likewise, V has a long-run stationary distribution with a mean of $E[V] = \alpha^2\gamma/\delta + \beta^2\eta/\nu$ and a variance of $Var[V] = \alpha^4\gamma/2\delta^2 + \beta^4\eta/2\nu^2$. Although the two processes for are interdependent, the unconditional limiting distribution of both variables is a sum of gamma distributions.

The fundamental theorem of asset pricing, see Duffie (1992) states that the price of a zero-coupon bond – as a function of the two state variables and the time to maturity – is equal to the following expectation:

$$G(r, V \mid \tau) := E^Q\left[e^{-\int_0^\tau r_s ds} \mid r_0, V_0\right], \tag{10}$$

where $E^Q[.]$ denotes the risk neutral (conditional) expectation with respect to the measure defined by equations (6) - (9). We refer the interest reader directly to Longstaff-Schwartz (1992) for a derivation of the solution to the corresponding Partial Differential Equation (PDE). This leads to:

$$G(r, V \mid \tau) = A(\tau)^{2\gamma} B(\tau)^{2\eta} exp\{\kappa\tau + C(\tau)r + D(\tau)V\} \tag{11}$$

where,

$$A(\tau) = \frac{2\phi}{(\delta+\phi)(\exp(\phi\tau)-1)+2\phi} \tag{12}$$

$$B(\tau) = \frac{2\psi}{(\nu+\psi)(\exp(\psi\tau)-1)+2\psi} \tag{13}$$

$$C(\tau) = \frac{\alpha\phi(\exp(\psi\tau)-1)B(\tau)-\beta\psi(\exp(\phi\tau)-1)A(\tau)}{\phi\psi(\beta-\alpha)} \tag{14}$$

$$D(\tau) = \frac{\psi(\exp(\phi\tau)-1)A(\tau)-\phi(\exp(\psi\tau)-1)B(\tau)}{\phi\psi(\beta-\alpha)} \tag{15}$$

and

$$\phi = \sqrt{2\alpha+\delta^2} \tag{16}$$

$$\psi = \sqrt{2\beta+\nu^2} \tag{17}$$

$$\kappa = \gamma(\delta+\phi)+\eta(\nu+\psi) \tag{18}$$

Despite the somewhat messy-looking set of equations, we *do* have a closed-form analytic expression for the zero-coupon bond price. This will be quite helpful when estimating the mortality-yield curve model

3.2 The Mortality Component

Parallel to the notion of zero-coupon bond (ZCB), we define a pure life endowment contract that pays-off one dollar conditional on surviving to a specific age. For example, the investor might pay \$0.30 at age 65, in exchange for a guarantee to receive \$1 at age 95, conditional on survival. Indeed, just like a coupon-bearing bond is a staggered collection of ZCBs, so to, an SPIA is a bundle of staggered pure life endowment contracts.

To reduce some of the clutter, we introduce shorthand notation for $G(\tau) := G(r, V \mid \tau)$ – and remember that it is an implicit function of six parameters and two state variables – and denote the price of a pure life endowment contract by:

$$L(x,\tau) := E^Q\left[e^{-\int_0^\tau (r_s+h_{x+s})ds} \mid r_0, V_0\right] = G(\tau)e^{-\int_x^{x+\tau} h_s ds}, \tag{19}$$

where h_s denotes the hazard rate at s – which is deterministic and independent of r_s – and $\exp\{-\int_x^{x+\tau} h_s ds\}$ is the probability of survival.

The actuarial price (pure premium) of any Single Premium Immediate Annuity (SPIA) is the sum of it's parts:

$$FA(x,N) = \left(\sum_{i=1}^{N} G(\frac{i}{12}) + \sum_{i=N+1}^{\infty} L(x,\frac{i}{12})\right), \tag{20}$$

where N denotes the number of guaranteed months. In practice, one has to pay $(1+l)FA(x,N)$ to obtain one-dollar-for-life, where l denotes the insurance loading for profits, commissions and fees[3]. In other words, $(1+l)FA(x,N)$ denotes the theoretical market price charged for a Single Premium Immediate Annuity (SPIA), issued to an individual at age x, that will pay \$1 per month for life, with a N-month payment certain guarantee.

It is somewhat more convenient to re-define a truncated hazard rate so that, using the Gompertz specification:

$$h^*_{x+t} = \begin{cases} 0 & t \leq N/12 \\ h_x e^{gt} & t > N/12 \end{cases}, \tag{21}$$

in which case we can re-write $FA(x,N)$ in equation (20) as:

$$FA(x,N) = (1+l)\sum_{i=1}^{\infty} F(\frac{i}{12})e^{-\int_x^{x+\tau} h^*_s ds}. \tag{22}$$

In our estimation, we will replace the integral $\int_0^\tau h^*_{x+s}ds$ with a discrete monthly sum. The upper bound of the summation sign will be truncated at the end of the mortality table, which is age 115.

4 Description of the Data

4.1 Interest Rates.

The data was provided by the Bank of Canada. They have used a cubic spline methodology to extract artificial zero-coupon bond prices from the coupon-bearing yield curve. Thus, for Canadian government bond yields, we have weekly observations of yields with maturities of 1-month, 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year and 30-year from January 2, 1985 to May 2, 2001. All bond yields are observed on Wednesday (or Thursday if Wednesday happens to be holiday) and represent (synthetic) zero coupon rates of corresponding maturities. Figure #1, in the appendix, displays the yield curve over the entire period.

4.2 Annuity Prices.

The annuity data – which covers the period 1985 to 2001 – was collected with the assistance of CANNEX Financial Exchanges Limited, a company that compiles payout rates from the most competitive Canadian insurance companies, and then sells the information (quotes) to brokers and financial planners.

[3]We ignore any fixed costs by absorbing them in the parameter l, since all the policies in our database have the same notional value of \$100,000.

The actual data file consists of two independent parts. The first is a 16-year time series of monthly payout rates from a Single Premium Immediate Annuity (SPIA) for a 65 year old (male and female) with a ten year guarantee, per C$100,000 tax sheltered funds. These funds must originate from a Registered Retirement Savings Plan (RRSP), or qualified plan. And, indeed, one obtains slightly more favorable payout rates from qualified plans, due to the possible mitigation of adverse selection[4]. The ten year guarantee implies that *even* if the annuitant dies – during the next ten years – payments will continue to the beneficiary, up to 10 year. These are currently the most common annuity contracts in Canada.

In any event, our database tracks *how much* life-time (monthly) income a 65 year-old would have received per $100,000 initial purchase, over the last 16 years. Figure #2 displays a sample of these quotes. It is the time series of the annuity payouts during the period 1985 - 2001, for both males and females at age 65.

A second part of the database provides the same information – starting only in October 2000 – but, for ages 55,60,65,70,75 and 80 and with 0,5,10,15,20 and 25 year guarantees. This portion of the database – which is also updated weekly – provides us with a cross-sectional term structure of mortality-contingent claims.

To be understand the structure of our data, Table #1 displays a sample quote for one of the companies in the database (Manulife Financial), on May 2, 2001.

Certain \ Age	m **55** f		m **60** f		m **65** f		m **70** f		m **75** f		m **80** f	
0 yrs	631	590	686	633	765	694	877	780	1039	911	1259	1096
5 yrs	628	589	681	631	755	689	855	770	989	887	1146	1036
10 yrs	620	584	666	623	726	674	799	741	879	825	940	901
15 yrs	607	578	644	611	687	652	729	699	764	745	774	765
20 yrs	591	569	618	596	643	625	662	651	673	668	665	664
25 yrs	573	559	589	578	601	594	608	605	610	609	N.A.	N.A.

Table #1: Sample quotes on May 2, 2001, annuity payouts per C$100,000 (RRSP)

For example, a 75 year-old female would be guaranteed to receive $911 for the rest of her life, per $100,000 initial premium. This annuity would have no guarantee period, which means that if she were to die one instant after purchasing the life annuity (technically it would have to be after the first payment) her beneficiaries or estate would receive nothing in return. The entire $100,000 would be lost, or more precisely, spread amongst the survivors to enhance their investment returns. Indeed, the $911 monthly income, for those who survive,

[4]Under current legislation, the Canadian Income Tax Act (ITA) stipulates that all Canadians, by age 71, must convert their Registered Retirement Savings Plan (RRSP) to a Registered Retirement Income Fund (RRIF) or purchase a non-reversible life annuity from an insurance company in order to maintain the funds in a tax-sheltered status.

consists of blended principal and interest, as well as other peoples money.[5] A male would receive slightly more, $1,039, due to the lower life expectancy.

A few things should be obvious from the table. First, the higher the purchase age, all else being equal, the greater the annuity income. Naturally, the life expectancy is shorter and the initial premium must be amortized over a shorter time period. Likewise, a longer guaranteed period yields a lower annuity income, because of the lack of mortality credits during the initial period. In fact, an 80 year-old requesting a 20 year guarantee will receive virtually the same amount ($664) as a male of the same age, since neither is likely to live past the 20 year certain period, and hence the annuity is essentially a collection of default-free zero coupon bonds. One final point worth noting is that Manulife Financial does did not quote annuities with 25 years certain, for an 80 year old, which is why the table contains N.A. This omission occurs in various places throughout the data, namely, certain companies do not quote certain products at certain times. From a statistical perspective – in creating our averages – we discarded the highest and lowest quotes and took an average of the remaining numbers. This, effectively, mitigates some of the credit risk concerns, as well the problem of sporadic omissions.

We believe that our data consists of every Canadian company offering annuities during the period 1985 - 2001. The number of companies varies over time – from 20 to 14 – as a result of merges, acquisitions and (yes) bankruptcies[6] over this period. Unfortunately, prior to October 2000, we only have access to monthly quotes for age 65 (male and female) with a 10-year certain period. In other words, only columns 5 and 6 and row 3, in the Table #1. From October 2000 and onward, we have the full panel, on a weekly basis.

Some stylized, perhaps obvious, facts emerge from a casual examination of the entire database:

(1) Payout rates fluctuate from month-to-month and are highly correlated with prevailing interest rates in the market. Over the 1985-2001 time frame, the highest (median across companies) payout rate was $1130 for males and $1060 for females, both in early 1985. Likewise, the lowest (median) payout rate was $800 for males and $900 for females, both in

[5] To remind the reader on how a life annuity functions, we provide the following example. There is a 20% chance that a 95 year old female will die in the next year, according to Statistics Canada Population Standard Life Tables: 1990-1992. Thus, if five such Canadian females enter into a one-year annuity (tontine) agreement, by investing $100 each in a pool yielding 5%, the funds will grow to $525 by year end. Of the starting five, four are expected to survive, leaving $525/4=$131.24 per survivor. This is a net (expected) return of 31.25%. This far exceeds the risk free return of 5% (or perhaps any risky return), because the annuitants have seceded control of assets in the event of death.

[6] Confederation Life was declared insolvent in 1994, and the company was seized by the Office of the Superintendent of Financial Institutions (OSFI). However, annuitants did not loose any money, mainly because the payments are guaranteed by the industry pool, COMCORP.

late 1993. This is a variation of 70% of possible retirement income.

(2) Males, 65 years old, obtain approximately 5% - 9% percent more consumption (monthly payout) than Females on average. In other words, females pay more for the same consumption stream. This markup is purely a function of female longevity. Indeed, during the last 16 years, this ratio has increased from about 5% to its current 9%. The upward trend is due to the general decline in interest rates over the same period, which makes the mortality effect more pronounced[7]. In fact, one can rigorously 'prove' that the derivative of the ratio of annuity factors, $a_x^f(r)/a_x^m(r)$, with respect to r, is negative.

(3) There are substantial benefits to searching as evidenced by the dispersion between the maximum and minimum quotes provided by various insurance companies. In any particular month, the best and worst payout quotes varied by more than 5%, per monthly payout, per C$100,000. Identical financial products, in a competitive market, do not usually display such wide pricing variations. Although credit issues play a small role, we suspect that the non-competitive quotes amongst the same credit risk-class, are an indication of an insurance company that is trying to "turn off" the annuity "tap".

To get a sense of the variation in annuity quotes at any given point in time, Figure #7, on the last page of this article, displays the relationship between credit rating, (from Moody's, Standard & Poors and A.M. Best), and the average relative payout from immediate annuities. Clearly, credit risk is a factor. However, Our empirical analysis will focus on the cross-section and time-series of *median payout rates*, by discarding one outlier on each side of the average quote. Therefore, being that we are focusing on median rates, and not specific companies, we are confident that credit issues have been mitigated.

In sum, we have monthly observations for both male and female of age 65, from January 10, 1985 to May 3, 2000 and weekly observations for both male and female of age 55, 60, 65, 70, 75, and 80 from October 10, 2000 to May 2, 2001. In addition, for the weekly data we have annuity prices with different years of guarantee (namely 0, 5, 10, 15, 20 and 25) for both male and female of each age group.

5 Estimation of the Model.

The estimation procedure works as follows. On a heuristic level, we seek to minimize the squared difference between the theoretical and observed value of the annuity factor. Let $\widehat{G}(r_t, V_t, \tau_i)$ denote the observed government bond price at time t with maturity τ_i and $\widehat{FA}_t(r_t, V_t, x, T_i)$ denote the observed annuity price at time-t for age x with guarantee (number of years) T_i, assuming that the insurance loading is constant at $100l\%$. To infer the parameter set in the zero-coupon bond price and the parameters of the mortality function

[7] We thank Mark Warshawsky for pointing this out to one of the authors.

as well as the insurance loading, we fit the government yield curve and the annuity prices simultaneously. The objective function is defined as the squared difference between theoretical treasury bond price and observed treasury bond price *plus* the squared difference between theoretical annuity price and observed annuity price, i.e.

$$\sum_{i=1}^{N}(G(r_t,V_t,\tau_i)-\widehat{G}(r_t,V_t,\tau_i))^2 \\ +C\sum_{x=\{55,60,65,70,75,80\}}\sum_{i=\{0,5,10,15,20,25\}}((1+l)FA_t(r_t,V_t,x,i)-\widehat{FA}_t(r_t,V_t,x,i))^2 \tag{23}$$

where τ_i is the maturity of government bond, N is the number of different maturities along the yield curve, and the constant $C(>0)$ is a scaling parameter to ease convergence of the optimization procedure. For each of the 36 weeks, we have 6*6=36 observations, which correspond to the six ages multiplied by the six guarantee periods.

First, we utilize the information contained in the weekly observations of government bond yields and annuity prices over the period of October 11, 2000 to May 2, 2001. During this period, we have weekly observations of the government bond yields with 10 different maturities and the annuity prices for different age groups, namely age 55, 60, 65, 70, 75 and 80, and for each group with different guarantees, namely 0-year, 5-year, 10-year, 15-year, 20-year and 25-year.

Each week, we back out the parameter values of the Longstaff-Schwartz (1992) bond pricing formula, together with the parameters of the mortality function, as well as the insurance load l. We note that the parameters of the mortality function are very robust over this period. Notwithstanding the above technicalities, following this procedure, we obtain in-sample estimates for the parameters. First, Table #2 contains the values of the interest rate parameters, $\{\alpha,\beta,\gamma,\delta,\eta,\nu\}$, together with their standard deviations.

	γ	δ	η	υ	α	β
Mean:	1.1905	0.1233	0.0689	5.9150	0.0092	3.5613
Std.Dev.	1.4967	0.1501	0.0965	2.2459	0.0121	0.6452

Table #2: Implied values from cross-section of bonds and annuities

We remind the reader that both r and V are estimated as well, however, each weekly datapoint produced it's own value. In other words, we have a time series of (changing) values for r and V. Also, the numbers in Table #2 are quite different from those estimated by Longstaff-Schwartz (1992). We suspect part of the difference is the Canadian versus U.S. environment, as well as the period over which the estimates were made. We are covering a small period in 2000 - 2001, while Longstaff-Schwartz (1992) covered the entire 1964 - 1989.

Finally, the implied hazard rate parameters were as follows.

Female	h_{55}	h_{60}	h_{65}	h_{70}	h_{75}	h_{80}	g	l
Mean	0.00174	0.00320	0.00578	0.01025	0.01835	0.03242	0.11637	0.02144
Std.Dev.	0.00021	0.00034	0.00059	0.00091	0.00141	0.00216	0.00650	0.01053
Male	h_{55}	h_{60}	h_{65}	h_{70}	h_{75}	h_{80}	g	l
Mean	0.00428	0.00710	0.01178	0.01641	0.03174	0.05111	0.10068	0.02040
Std.Dev.	0.00079	0.00112	0.00164	0.00237	0.00343	0.00469	0.01788	0.01060

Table #3: The implied hazard rates from annuity prices: Oct. 2000 - May 2001

Table #4 compares the probability of survival to age 85 – conditional on being alive at age 55 – using the estimated parameter values during the period October 2000 to May 2001 in comparison to the probabilities obtained from the Individual Annuity Mortality (basic) 1996 compiled by the U.S.-based Society of Actuaries. (Table #1) Thus, we see that the implied values and the mortality table values are relatively close to each other.

$x = 55$, survives to:	60	65	70	75	80	85
Female: $(0.00174, 0.11637)$	0.988	0.967	0.931	0.870	0.771	0.621
Male: $(0.00428, 0.10068)$	0.972	0.928	0.860	0.758	0.616	0.436

Table #4: Implied survival probabilities using Gompertz specification.

Interestingly, the implied survival probabilities for males are slightly lower than those provided in Table #1, and the survival probabilities for females are slightly higher for the earlier ages. Nevertheless, for females they match two significant digits. Furthermore, they are much closer to the 1996 IAM table, in comparison to the 1983 IAM table.

5.1 Evolution of Mortality Over Time.

Whereas the previous section looked at the implied average parameters over the (shorter) period October 2000 to May 2001, this section examines those values during the entire period of 1985 to 2001. As we mentioned previously, limited data was available on a monthly basis, and only for ages 65 (m/f) with a 10 year period certain. With only one age, we are not able to imply unique values for both h and g in the Gompertz specification. Therefore, we decided to fix the values of g as estimated in the later period – i.e. $g = 0.116$ for females and $g = 0.100$ for males – and then imply the value of h_{65} over time. We thus obtain a time series of instantaneous hazard rates for males and females aged 65, assuming the gompertz specification for the probability of survival.

Although we only have one data point for annuity prices for each of those months, we stress that this number is an average across all insurance companies offering annuities in Canada during that time period. Also, we have the entire yield curve for each of those months.

Thus, we implied the instantaneous interest rate, volatility and hazard rates using the same procedure and objective function described in section 5. Figure #3 displays the implied instantaneous interest rate, figure #4 displays the implied instantaneous volatility of interest rates, figure #5 displays the implied instantaneous female hazard rate at age 65, and figure #6 does the same for males.

It is quite obvious from both figures #5 and #6 that the implied hazard rates are not very stable of smooth over time. There is a definite negative trend, but it is punctuated by spikes and relatively high volatility. We conjecture that part of the variation comes from a lag in responding to changes in the yield curve. Stated differently, on any given day, the insurance company does not use the entire yield curve to provide annuity quotes. Rather, they use a 'scalar discount rate' which relates to (capital charges and) what the company expects to earn on the assets in their general account. This number may, or may not, change based on the current yield curve. Furthermore, the reduction in the volatility of the hazard rate – during the last 16 years – might reflect a movement towards more instantaneous adjustments to changes in the yield curve and more efficient pricing *vis a vis* the risk-free yield curve.

6 Conclusion

Most of the existing research on corporate credit risk and default, is plagued by sparse data sources and complex estimation problems. Even when data is available, one is never certain of the appropriate default probabilities, and the recovery rates. Single Premium Immediate life Annuities (SPIA) are the purest form of defaultable coupon-bearing bonds. They provide us a with a unique laboratory for testing various theories about the term structure of defaultable interest rates.

We have compiled a 16-year cross-sectional database of annuity payouts and have used them to 'calibrate' a Longstaff and Schwartz (1992) model. We use the data to imply the market's expectations about default and examine the properties over time.

Our unique methodology confirms that (i) Canadian mortality rates have been declining over the last 16 years, (ii) the profitability (or mark-up) for annuities is on the order of 2% - 3%, (iii) the annuity market does not instantaneously adjust to a change in the yield curve, and (iv) a Longstaff-Schwartz (1992) specification provides a reasonable fit to the term structure of mortality-contingent claims.

A natural extension of this research is to use our Longstaff-Schwartz-Gompertz (LSG) model to price contingent claims on an underlying mortality-contingent claim.

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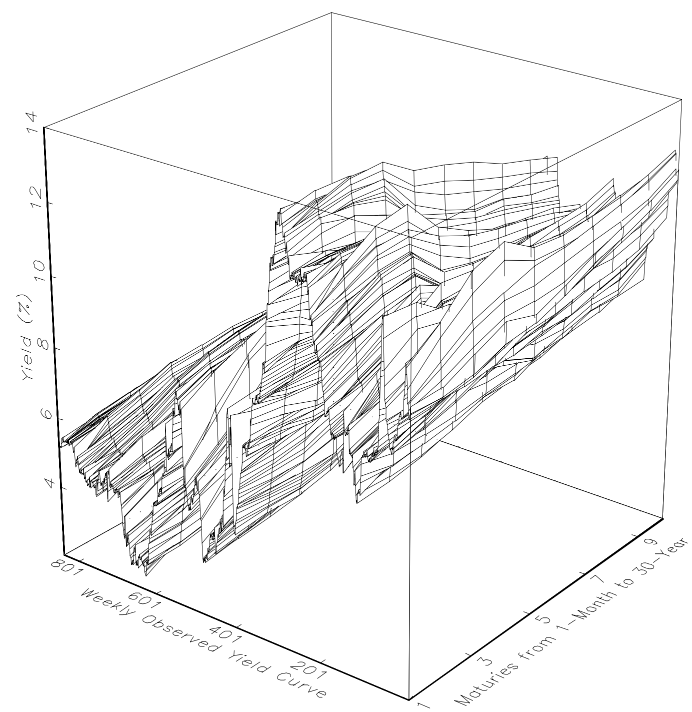
Figure 1. Historical Canadian T-Bill and Bond Yields (85.1.2--01.2.14):

with maturies 1-, 3-, 6-month, 1-, 2-, 3-, 5-, 7-, 10-, and 30-year



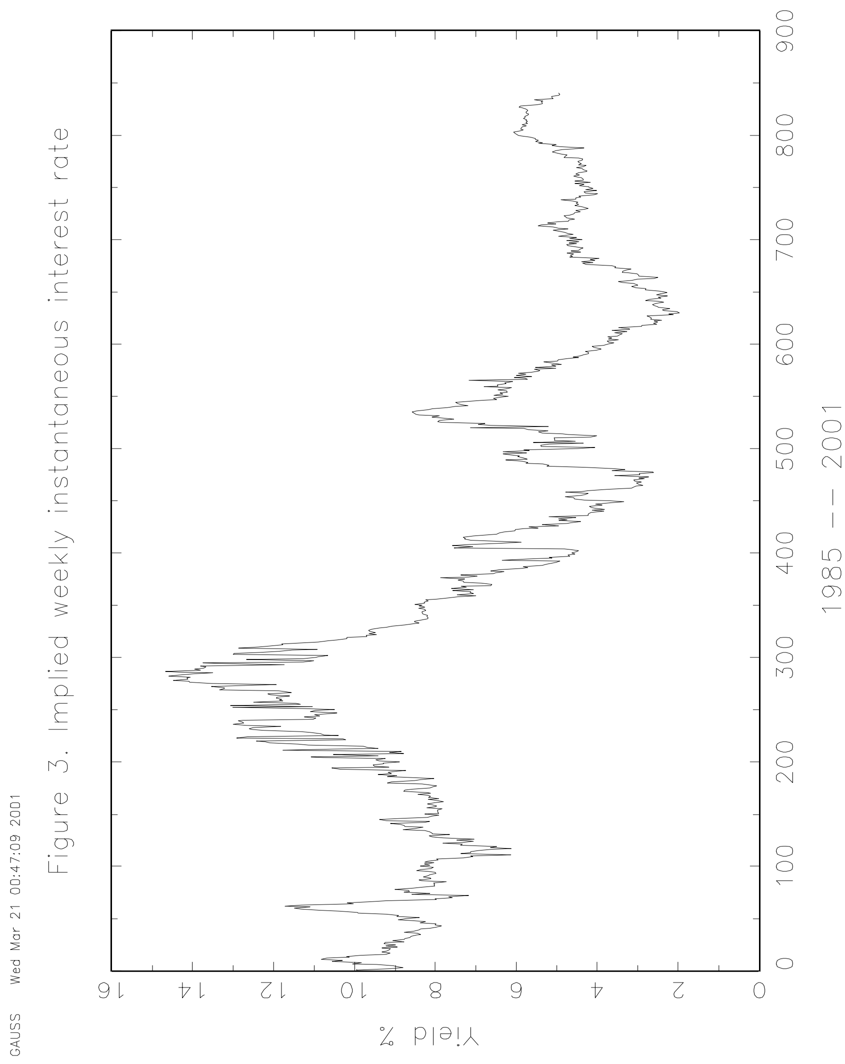


Figure 3. Implied weekly instantaneous interest rate

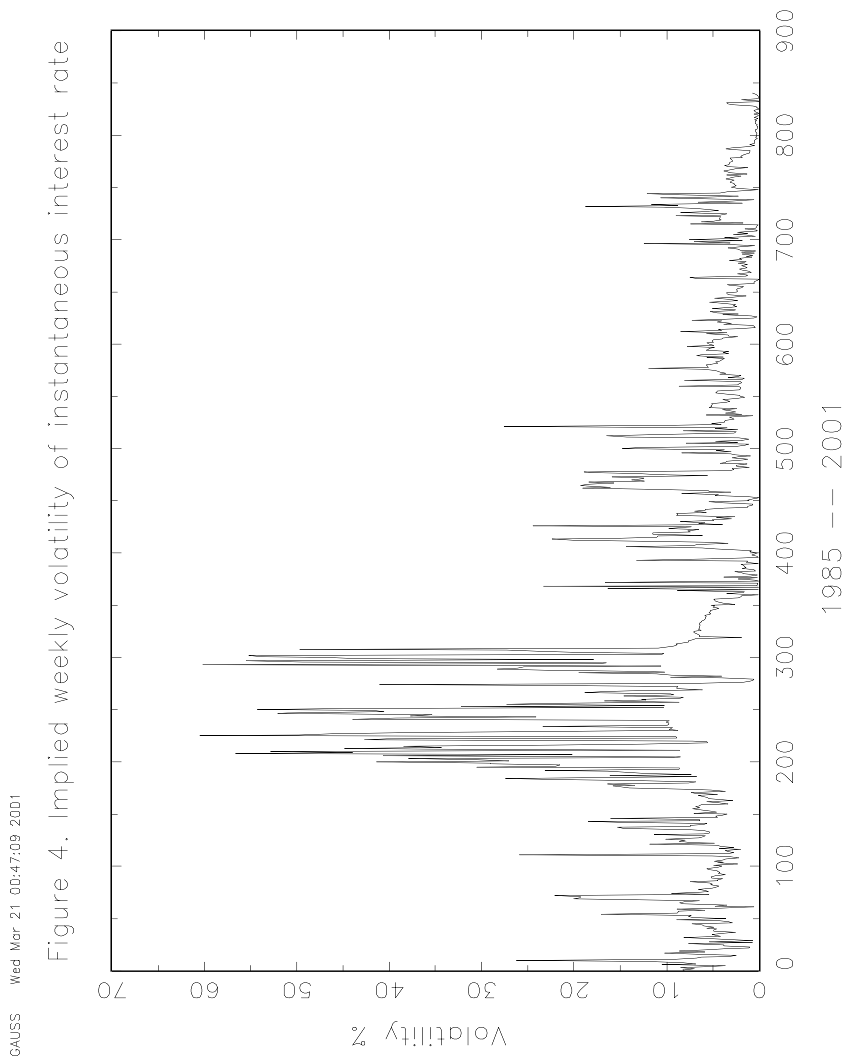

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Figure 4. Implied weekly volatility of instantaneous interest rate
Volatility %
1985 -- 2001
0
10
20
30
40
50
60
70
0
100
200
300
400
500
600
700
800
900

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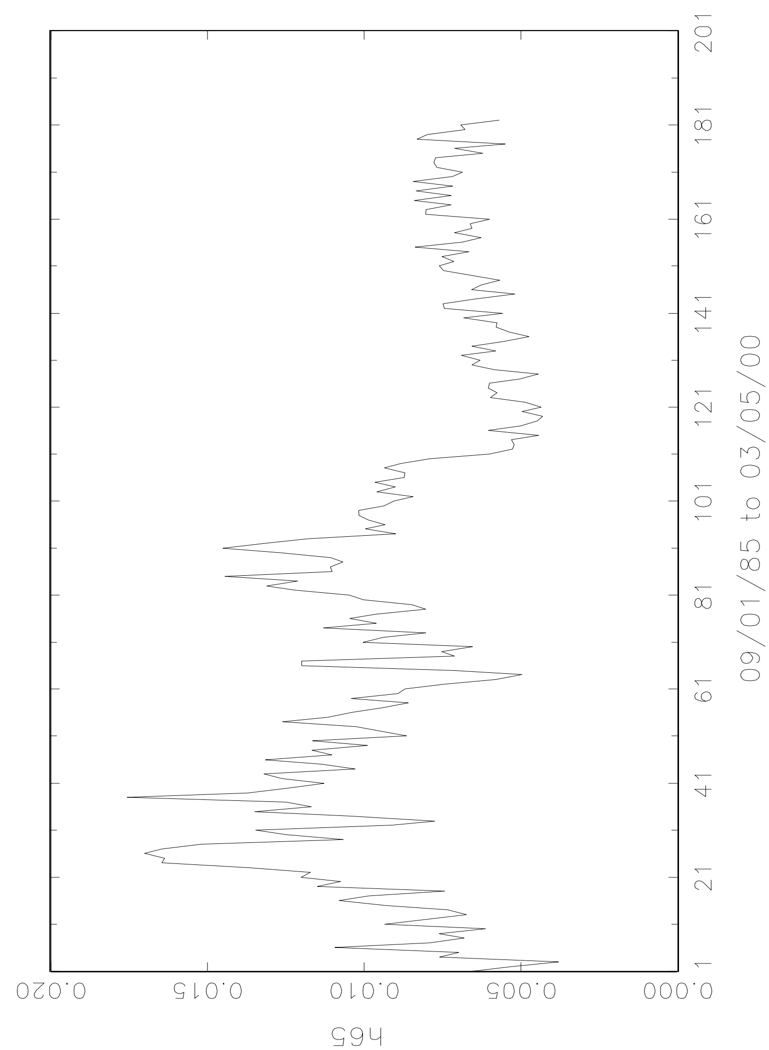
Figure 5. Implied mortality function parameter for female age 65 (h65)



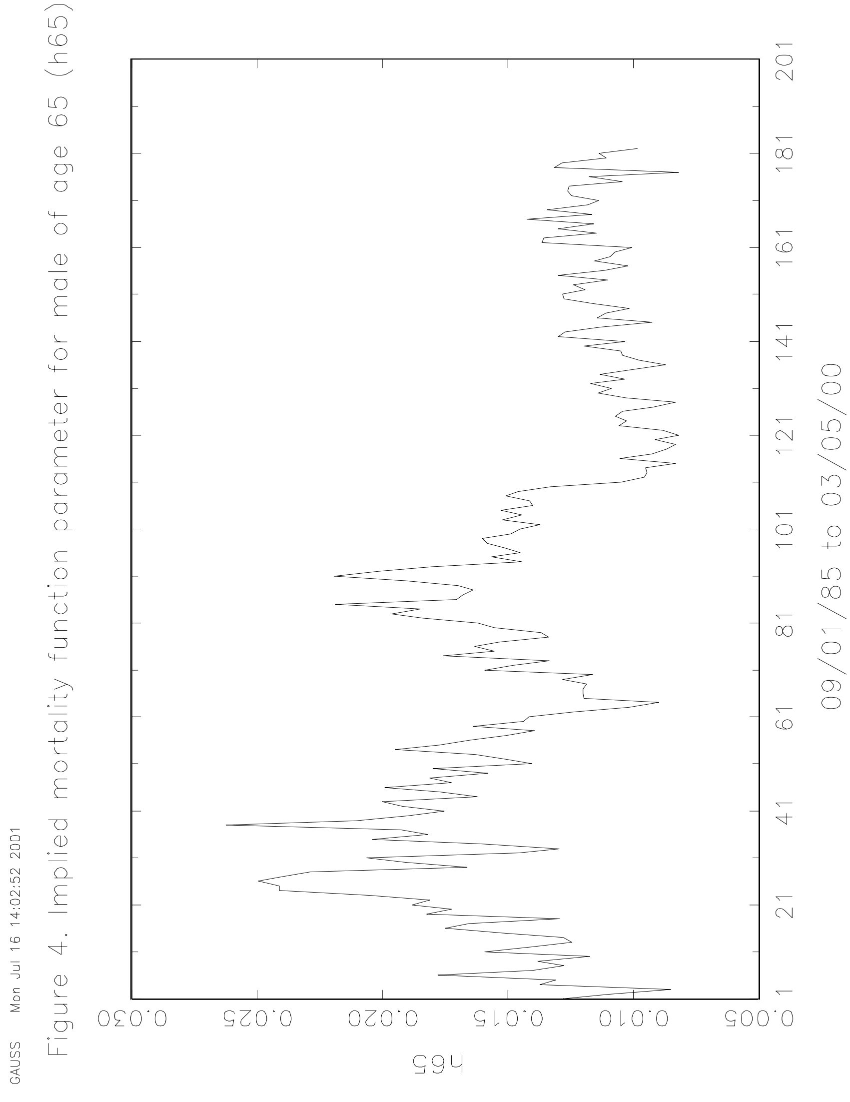


Figure 4. Implied mortality function parameter for male of age 65 (h65)