

**SPENDING RETIREMENT ON PLANET VULCAN:
*The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates***

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Abstract:

In this paper we describe the optimal retirement spending policy for a utility-maximizing consumer (i.e. Mr. Spock) facing a stochastic lifetime. We contrast optimal behavior with recommendations offered by the investment media and financial planners, for example that a retiree spend approximately 4% of their retirement nest egg each year adjusted for inflation, or similar heuristics. Methodologically we solve a Keynes-Ramsey (1928) model using the Calculus of Variations (CV) and calibrated to inflation-adjusted returns, pre-existing pensions income and a Gompertz law of mortality. Solutions are available in closed form.

Our main practical conclusion is that counseling retirees to set initial spending at constant 4% of their nest egg is consistent with lifecycle theory only under a limited set of mortality-risk aversion and time preference parameters. We show exactly how the optimal behavior in the face of personal longevity risk is a plan that adjusts consumption downward in proportion to survival probabilities – adjusted for pension income and risk aversion -- as opposed to blindly withdrawing the same inflation-adjusted income for life.

Our framework enables us to clearly illustrate the impact (and benefit) of pension annuities and longevity insurance on the optimal plan. We hope that our simplified approach – which truly goes back to first principles -- will help create a common language and improve the dialogue between financial economists and the financial planning community. The chasm continues to grow and the stakes in this matter are too high for the scholarly community to sit by idly.

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[1.] Introduction:

"...The first problem I propose to tackle is this: how much of its income should a nation save?..." With these immortal words the 24-year-old Cambridge University economist Frank R. Ramsey began what has become a renowned paper published in the *Economic Journal* two years before his tragic death in 1930. The so-called Ramsey (1928) model elucidated in that paper and the resultant Keynes-Ramsey rule, implicitly subsumed by thousands of economists in the last 80 years -- including Fisher (1930), Modigliani (1984, 1956), Phelps (1962) and eventually Yaari (1965) -- represents the foundation for the mathematical framework for consumption and utility optimization. It is also the workhorse supporting the original asset allocation models of Samuelson (1969) and Merton (1971).

In its basic form the normative lifecycle model (LCM) -- now formulated as a hypothesis -- assumes that a rational individual seeks to maximize, over all admissible consumption paths, the discounted additive utility of consumption over their entire life. The Calculus of Variation (CV) model -- first expressed mathematically by Ramsey (1928) -- is studied by most graduate students of economics and has been extended by hundreds if not thousands of scholars since. Ask a first-year graduate student in economics how a consumer should be "spending" their capital over time and most likely they will respond with a Ramsey-type model spreading wealth between time zero and terminal time T.

Of course, the literature has advanced since 1928. Later versions of (normative) LCM allow for bequest motives, a family of consumers, multiple investment asset classes, dates of death and disability -- all driven by stochastic mortality rates, stochastic term structures, stochastic wages correlated with equity markets, inflation dynamics, etc. Advanced versions of the LCM are no longer solved using the Calculus of Variations (CV), but instead employ sophisticated Dynamic Programming algorithms which often lead to additional control variables besides consumption, such as optimal investment asset allocation, insurance purchase rates, annuitization, etc. The growing literature -- which often falls under the title of "portfolio choice" or extensions of the Merton model -- is wide and deep. We count over 50 scholarly articles on this topic published in the top five academic journals in finance just over the last decade.

Unfortunately, the financial planning community has ignored most of these rich models. Nowhere is this more evident than in the "retirement income planning" world.

Alas, the financial crisis of the last few years coupled with the growing literature in behavioral economics has moved the practice of personal finance farther away from adopting an optimization approach to planning. Thus today, many of the popular products used and strategies employed by

individuals in their portfolios are at odds with financial economics. See the articles by Bodie and Treussard (2007), Kotlikoff (2008) as well as the monograph edited by Bodie, McLeavey and Siegel (2008). This raises concerns about conventional financial planning as practiced in the 21st century when compared with lifecycle models of investment and consumption.

Part of the issue is that many of these scholarly articles are not geared towards practitioners, nor are they written in the practitioner's language. The objective in most of the recent research is to (i.) formulate the hypothesis, and (ii.) generate testable implications, and then (iii.) take the model to the data and test it. Other strands of the more mathematically-oriented literature have focused on proving existence and uniqueness theorems, or general classifications²; as opposed to deriving new results or applicable insights. And while these efforts advance the science of financial economics, much of the literature in the field seems to have lost sight of the Ramsey-inspired question: *How much should you save vs. consume?*

So, in this paper we investigate the advice offered by the financial planning community as it pertains to retirement spending policies vis-à-vis the "advice" of financial economists using a rational utility-maximizing model of consumer choice³.

In particular we focus exclusively on the impact of lifetime uncertainty – often called longevity risk – on the optimal consumption and spending policy. We calibrate the model to real (risk-free) interest rates and actuarial mortality rates so that we can compare our results with advice being dispensed by the popular media and financial planning community. Our contribution isn't in the modeling, which is decades-old, but in the numerical calibration, results and contrasting with common financial wisdom.

[1.1] Context:

Within the community of retirement income planners, an often-referenced paper is the work by Bengen (1994) in which he used historical (Ibbotson Associates) equity and bond returns to search for the highest allowable spending rate that would sustain a portfolio for 30 years of retirement. Using a 50/50 equity/bond mix, Bengen settled on a spending rate between 4% which has now attained folklore status in the popular media.

² See for example the classic Richard (1975) paper, or more recently the work by Bodie, Detemple, Ortuba and Walter (2004), or Babbal and Merrill (2006).

³ Hence our use of "Planet Vulcan" in our title, inspired by Thaler and Sunstein who distinguish "humans" from perfectly rational "econs," much like Spock in TV's Star Trek, who originates from the Planet Vulcan.

Here is a direct quote from Bengen (1994):

“...The withdrawal dollar amount for the first year -- calculated as the withdrawal percentage times the starting value of the portfolio -- will be adjusted up or down for inflation every succeeding year. After the first year, the withdrawal rate is no longer used for computing the amount withdrawn; that will be computed instead from last year's withdrawal, plus an inflation factor....”

It is hard to over-estimate the influence of this article and its embedded “rule” on the contemporary practice of retirement income planning. Other widely-quoted studies in the same vein include an article in the *Journal of the American Association of Individual Investors* (AAII), by Cooley, Hubbard, and Walz (1998) called the Trinity Study amongst practitioners. By our count these and related studies have been quoted thousands of times in the last two decades (Money Magazine, USA Today, Wall Street Journal)⁴. The 4% spending rule now seems destined for the same immortality enjoyed by other rules of thumb such as buy-term-and-invest-the-difference, or dollar-cost-averaging. And, while numerous authors (most within the financial planning community) have extended, refined and re-calibrated these spending rules, their spirit remains intact across all versions⁵.

And yet this “*start by spending x%*” strategy has absolutely no basis in economic theory. We are not the first to point this out. For example, a paper by Sharpe, Scott and co-authors (2007, 2009) raised similar concerns and alluded to the need for a life-cycle approach, but never actually solve or calibrate such a model. So, let us be clear: We don't seek to discredit or ridicule the widely-adopted 4% approach. In fact, our hope is to formulate a model of consumer choice from very first principles, à la Ramsey (1928) to see if there is any common ground. More importantly, we want to demonstrate how longevity-risk aversion -- in contrast to financial risk aversion, so familiar to financial analysts – impacts retirement spending rates.

A number of recent articles have teased-out the implications of mortality and longevity risk on portfolio choice and asset allocation, for example the paper by Chen, et. al. (2006) in the *Financial Analysts*

⁴ Money Magazine, August 16, 2007 article by Walter Updegrave, or CBS MarketWatch, July 29, 2009 article by Janet Kidd Stewart

⁵ In fact, we (the authors) are partially guilty of helping proliferate this approach by deriving analytic expressions for the portfolio ruin probability assuming a constant consumption rate.

Journal (FAJ), but in this paper we want to focus exclusively on longevity risk as it relates to retirement income planning.

The remainder of this paper is organized as follows. Section [2.] provides a high-level analytic summary of the lifecycle model (LCM) and the implied optimal consumption rates as they relate to longevity risk-aversion. Section [3.] provides extensive numerical results and illustrations for a number of cases including pre-existing pension income. Section [4.] provides a qualitative summary of our main insights. All exhibits are displayed after the bibliography and references are at the end of the paper.

[2.] The Ramsey-Modigliani-Yaari Lifecycle Model⁶

The objective value function within the lifecycle model (LCM) during the retirement years when labor income is zero, assuming no bequest motive, can be written as follows:

$$\max_c V(c) = \int_0^D e^{-\rho t} ({}_t p_x) u(c_t) dt, \quad (\text{eq.1})$$

The variable x denotes the current age of the retiree, which is time zero when the entire consumption/spending plan is formulated. The parameter D , the upper bound of the utility integration, represents the maximum possible lifespan years in retirement. Note that results are insensitive to whether this number is taken as 50, 60 or even 100 years, since the probability of surviving to that time is quite small. The parameter ρ denotes the subjective discount rate (SDR), a.k.a. personal time preference, which is understood to be in the vicinity of the real long-term interest rate in the economy. The function $({}_t p_x)$, which uses the actuarial community's favored notation, denotes the conditional probability of survival from retirement age (x) to age ($x + t$). In this paper we parameterize $({}_t p_x)$ based on the so-called Gompertz law of mortality under which the biological hazard rate is: $\lambda_t = (1/b)e^{(x-m+t)/b}$ which grows exponentially with age. The 2 free parameters in our mortality model are \underline{m} which denotes the modal value (for example 80 years) and \underline{b} which denotes the dispersion coefficient (for example 10 years) of the future lifetime random variable. Both of these numbers will be calibrated to (objective) U.S. mortality tables and will be explained later in the numerical section. For now, think mean and standard deviation of longevity risk.⁷

⁶ Those readers who are not interested in the model derivation can skip directly to the results in section 3.

⁷ Note that in this paper we are not considering the possible (double) uncertainty within the hazard rate λ_s itself. This is often called stochastic mortality models, which leads to its own set of problems. We assume that the entire mortality term structure is known at retirement.

In our paper the utility function of consumption is assumed to exhibit constant Elasticity of Inter-temporal Substitution (EIS), which is synonymous with (and the reciprocal of) constant relative risk aversion (RRA) under conditions of perfect certainty. The exact specification we use is: $u(c) = c^{1-\gamma}/(1 - \gamma)$, where γ is the coefficient of relative risk aversion which can take on values from Bernoulli ($\gamma=1$) up to possibly infinity. Since we don't have any financial market uncertainty in our model, the risk we refer to is longevity risk and (γ) measures the aversion to this uncertainty.

One final ingredient needed for our analysis is the “actuarial present value” function denoted by $a_x^T(v, m, b)$, which depends implicitly on the survival probability curve (${}_t p_x$) via the parameters (m, b). It is defined and computed using the following.

$$a_x^T(v, m, b) = \int_0^T e^{-vs} ({}_s p_x) ds. \tag{eq.2}$$

Some readers will recognize equation (#2) as the retirement-age “price” – under a constant discount rate v -- of a life-contingent pension annuity that pays \$1 per year until the earlier of death and time T . When there is no mortality-risk, the value of equation (#2) collapses to the present value of an annuity.

Luckily, a closed-form representation of equation (#2) is possible in terms of the incomplete Gamma function $\Gamma(A, B)$, which is actually available in Excel⁸.

$$a_x^T(v, m, b) = \frac{b\Gamma(-vb, \exp\{\frac{x-m}{b}\}) - b\Gamma(-vb, \exp\{\frac{x-m+T}{b}\})}{\exp\{(m-x)v - \exp\{\frac{x-m}{b}\}\}}. \tag{eq.2a}$$

Going back to the problem at hand, the wealth trajectory (financial capital during retirement) is denoted by F_t and the dynamic constraint in our model – to go with the objective function equation (#1) -- can now be expressed as follows:

$$\dot{F}_t = v(t, F_t)F_t - c_t + \pi_0, \tag{eq.3}$$

where the dot on top is shorthand notation for a derivative of wealth (financial capital) with respect to time, π_0 denotes the income (in real dollars) from any pre-existing pension annuities and the function multiplying wealth itself is defined by:

⁸ We refer the interested reader to the book by Milevsky (2006) for a full discussion of the Gompertz law of mortality, its analytic representation and a derivation of equation (#2), which is well beyond the scope of this paper.

$$v(t, F_t) = \begin{cases} r, & F_t < 0 \\ R + \lambda_t, & F_t \geq 0 \end{cases} \quad (\text{eq.3a})$$

where $R \geq r$. The discontinuous function $v(t, F_t)$ denotes the interest rate on financial capital and allows F_t to be negative. In other words the interest rate charged on borrowings (if and when wealth is negative) is much higher than the riskless investment return. Credit cards and/or other unsecured lines of credit would be a good example of a situation in which $v(t, F_t) = R + \lambda_t$. The borrower pays R plus the insurance (to protect the lender in the event of the borrowers death).

Note that we do not assume a complete liquidity constraint that prohibits borrowing in the sense of Deaton (1991), Leung (1994) or Butler (2001) for example. What we don't allow is stochastic returns. This is essentially the Yaari (1965) set-up under which pension annuities are available, but not tontine annuities (which are still unavailable, 45 years after his original paper.) Once again, borrowers must buy life insurance to protect against default (death).

The initial condition is $F_0 = W$, where W denotes the investable assets at retirement. The terminal condition is that: $F_\tau = 0$, where τ denotes the wealth depletion time (WDT) at which point only the pension annuity income is consumed. The existence of a WDT was discovered within the economics literature, and has been explored by Leung (1994, 2007) in a series of theoretical papers. Of course, at this early stage of the process we don't yet know the exact value of the WDT, as we must solve for it. We will explain how later. In theory the WDT can be at the final horizon time $\tau = D$, if the pension income is minimal (or zero) and/or the borrowing rate is relatively low. To be very precise here, it is possible for $F_t < 0$ for some time $t < D$. We are not talking about the zero values of the function. Rather, the definition of our WDT is that: $F_t = 0; \forall t > \tau$, which is permanently. One can actually show that when $R > \rho$ the then by $\tau < D$. In our numerical results we assume this.

To recap, here is what constraint equation (#3) is expressing, in words this time: The rate of change in the financial trajectory – assuming it is still changing -- consists of investment gains minus consumption plus pension income, if any. After the wealth depletion time (WDT) we have $c_t^* = \pi_0, F_t = 0$.

On to the solution: The Euler Lagrange Theorem (ELT) from the Calculus of Variations leads to the following result⁹. optimal trajectory F_t in the region over which it is positive and assuming $v(t, F_t) = r$, can be expressed as the solution to the following second-order non-homogenous differential equation:

⁹ The technical derivation and proof is available in a working paper by Huang and Milevsky (2010)

$$\ddot{F}_t - (k_t + r)\dot{F}_t + rk_t F_t = -\pi_0 k_t, \quad (\text{eq.4})$$

where the double dots denote the second derivative with respect to time and the time dependent function: $k_t = (r - \rho - \lambda_t)/\gamma$ is introduced to simplify notation. The real interest rate r is a positive constant and a pivotal input to the model. Once again we reiterate that equation (#4) is only valid until the wealth depletion time τ . However, one can always force a wealth depletion time $\tau < D$ by assuming a minimal pension annuity $\pi_0 = \varepsilon$, as well as a large enough (arbitrary) interest rate $v(t, F_t)$ on borrowing when $F_t < 0$.

Moving on, the solution to the differential equation (#4) is obtained in two stages. First the optimal consumption rate while $F_t > 0$ can be shown to satisfy the equation:

$$c_t^* = c_0^* e^{kt} ({}_t p_x)^{1/\gamma}, \quad (\text{eq.5})$$

where $k = (r - \rho)/\gamma$ and the unknown initial consumption rate c_0^* will (soon) be solved for. The optimal consumption rate declines when the subjective discount rate ρ is equal to the interest rate r and hence $k = 0$. This is a very important implication (and observable result) from the lifecycle model. It is rational to plan to reduce one's standard of living with age, even if $(\rho = r)$. This is not because retirement expenses will be lower, or "you don't need as much".

Note also that consumption as defined above includes the pension annuity income π_0 . Therefore, the portfolio withdrawal rate (PWR) which is the main item of interest in this paper, is $(c_t^* - \pi_0)/F_t$ and the initial PWR (a.k.a. retirement spending rate) is $(c_0^* - \pi_0)/F_0$.

The optimal financial capital trajectory (also only defined until time $t < \tau$) which is the solution to equation (#4), can be expressed as a function of c_0^* as follows:

$$F_t = \left(W + \frac{\pi_0}{r}\right) e^{rt} - a_x^t(r - k, m^*, b) c_0^* e^{rt} - \frac{\pi_0}{r}, \quad (\text{eq.6})$$

where the modified modal value in the annuity factor is: $m^* = m + b \ln \gamma$. The actuarial present value term multiplying time-zero consumption in equation (#6) values a life-contingent pension annuity under a shifted modal value of: $m + b \ln[\gamma]$ and shifted valuation rate of: $r - (r - \rho)/\gamma$ instead of r . It has no economic interpretation other than being an intermediate step in our solution. However, plugging equation (#6) into the differential equation (#4) will confirm the solution is correct and valid over the domain: $t \in (0, \tau)$.

In other words, the value function in equation (#1) – and hence lifecycle utility -- is maximized when the consumption rate and the wealth trajectory satisfy equation (#5) and (#6) respectively. Of course, these two equations are functions of two “unknowns” c_0^* , τ , and we now must solve for them. We do this sequentially.

First, from equation (#6) and the definition of the wealth depletion time: $F_\tau = 0$, we can solve for the initial consumption rate:

$$c_0^* = \frac{\left(W + \frac{\pi_0}{r}\right) e^{r\tau} - \pi_0/r}{a_x^\tau (r - k, m^*, b) e^{r\tau}}. \quad (\text{eq.7})$$

Notice that when $\gamma = 1$ and $\pi = 0$ and $\rho = r$ the entire expression (#7) collapses to W/a_x^τ

Finally, the wealth depletion time τ is obtained by substituting equation (#7) into equation (#5) and searching the resulting non-linear equation over the range $(0, D)$ for the value of τ that solves $c_\tau^* - \pi_0 = 0$. In words, if a wealth depletion time exists, then for consumption to remain smooth at that point – which is actually a foundation of lifecycle theory -- it must converge to π_0 .

Mathematically the wealth depletion time (τ) satisfies the equation:

$$\frac{\left(W + \frac{\pi_0}{r}\right) e^{r\tau} - \frac{\pi_0}{r}}{a_x^\tau (r - k, m^*, b) e^{r\tau}} e^{k\tau} (\tau p_x)^{1/\gamma} = \pi_0, \quad (\text{eq.8})$$

In other words:

$$\tau = f(\gamma, \pi_0 | W, \rho, r, x, m, b) \quad (\text{eq.8a})$$

We are done. And although these eight equations all seem rather tedious, the important outcome is that the optimal consumption policy (described by equation #5) and the optimal trajectory of wealth (described by equation #6) are now available explicitly. Let us be clear however that the optimal consumption policy is *conditional* on the information available at time zero (retirement).

Practically speaking the wealth depletion time $\tau \leq D$ is extracted from equation (#8) and then the initial consumption rate is obtained from equation (#7). Everything else follows. Moreover, these expressions can be coded-up in Excel in just a few minutes. The next section provides numerical examples and illustrates the point of all this.

[3.] Numerical Examples and Cases

Our approach forces us to specify a real (inflation-adjusted) investment return, which requires some careful thought given that we are not assuming any uncertainty or variance of the return itself. On our planet Vulcan only life-spans are random. To that end we examined the real yield from U.S. TIPS over the last ten years, based on data compiled and reported by the Federal Reserve. The maximum real yield over the period was 3.15% for the 10-year bond, and 4.24% for the 5-year bond. The average real yield was 1.95% and 1.50% respectively for the 10-year and 5-year bond. The longer maturity TIPS exhibited higher yields, but which obviously entails some duration risk. So, after much thought we decided to use a real interest rate assumption of 2.5% for most of the numerical examples displayed in this paper's tables. Some additional examples are also provided assuming a (more optimistic) 3.5% real investment return. These (lower) numbers are consistent with a view expressed by Arnott (2004) regarding the future of the equity risk premium (ERP) as it relates to endowment spending rates.

As far as longevity risk is concerned, we assume that the retiree's remaining lifetime (random variable) follows the Gompertz law of mortality, calibrated to pension (RP2000) mortality tables. The Gompertz law states the mortality (hazard) rates increase exponentially over time. Thus, in most of our numerical examples we assume there is an 86.6% probability that a 65-year-old will survive to the age of 75, a 57.3% probability of reaching 85, a 36.9% probability of reaching 90, a 17.6% probability of reaching age 95 and a 5% probability of reaching 100. Note that we do not plan for certain life expectancy. Rather, we account for the entire term structure of mortality.

In terms of risk-aversion parameters, we use and display the results for a range of values. We show results for a retiree with a very low ($\gamma=1$) coefficient of relative risk aversion (CRRA), and a relatively high ($\gamma=8$) coefficient of relative risk aversion. To better understand the intuitive properties of these values we offer the following analogy to classical asset allocation models. An investor with a CRRA value of ($\gamma=4$) would invest 40% of his or her assets in an equity portfolio and 60% in a bond portfolio, assuming the equity risk premium is 5% and its volatility is 18%. Of course, our model does not have a risky asset nor does it require an equity risk premium (ERP), but the idea here is that one can in fact map the CRRA into more easily understood risk attitudes. Along the same lines, the very low (Bernoulli) risk aversion value of ($\gamma=1$) would lead to an equity allocation of 150%, and a high risk aversion value of ($\gamma=8$) implies an equity allocation of 20%. In fact, one of the main takeaways from this paper is to focus attention on the impact of risk aversion on the optimal portfolio withdrawal rates (PWR).

Finally, to round out the list of required economic (preference) parameters, we assume that the subjective discount rate (ρ) is set equal to the risk-free rate (mostly 2.5%). To those familiar with the basic life cycle model, this implies that the optimal consumption rate would be constant over time in the absence of longevity risk considerations. In the language of economics, when the reciprocal of the Elasticity of Inter-temporal Substitution (EIS) is equal to the Subjective Discount Rate (SDR), then according to the Modigliani lifecycle hypothesis, a rational consumer would spend the present value of their human capital evenly and in equal amounts over time.¹⁰

The question is: what happens when lifetimes are stochastic?

TABLE #1: OPTIMAL RETIREMENT CONSUMPTION AS FUNCTION OF AGE AND ASSUMED RETURN			
"MEDIUM" RISK AVERSION LEVEL (CRRA = 4)			
Age	Real Return = 1.5%	Real Return = 2.5%	Real Return = 3.5%
65	\$3.941	\$4.605	\$5.318
70	\$3.888	\$4.544	\$5.247
75	\$3.802	\$4.442	\$5.130

Note: 5% survival probability to age 100. Equivalent to Gompertz (m=89.335,b=9.5). No pension income.

We are now ready for some results. Assume a 65-year-old with a (standardized) \$100 nest egg. Initially we allow for no pension annuity income and therefore all consumption must come from the investment portfolio, which is earning 2.5% real per year. In this case the Wealth Depletion Time (WDT) is at the very end of the lifecycle (age 120). According to equation (#5) the optimal consumption rate at retirement age 65 is **\$4.605** when the risk aversion parameter is set to ($\gamma=4$), and the optimal consumption rate is **\$4.121** when the risk aversion parameter is set to ($\gamma=8$).

Notice that these numbers are within the range of numbers quoted by the financial planning community and the popular press for optimal withdrawal rates. At first glance this seems to suggest that their simple rules of thumb are (remarkably) consistent with rational economic lifecycle theory. Table #2 displays the consumption rates at age 65 as a function of an assumed interest rate that 100 basis points higher or lower. These numbers also seem to fall within the ballpark of conventional advice.

¹⁰ . For those interested in more information about parameter estimates for the EIS, we refer them to the paper by Hanna, Fan and Chang (1995).

Unfortunately the euphoria is short-lived. Namely, it is only in the first year of so-called retirement in which these numbers (might) coincide. As we explained after equation (#5) in the technical section, as the retiree ages they rationally consume less each year— in proportion to their survival probability adjusted for risk aversion. For example, in our baseline intermediate ($\gamma=4$) level of risk aversion, the optimal consumption rate drops from \$4.608 at age 65, to \$4.543 at age 70, then \$4.442 at age 75, then \$3.591 at age 90 and \$2.177 at age 100, assuming the retiree is still alive.

In other words, the rational (i.e. prevailing on Planet Vulcan) policy is to plan to spend less as you progress through retirement. It is suboptimal to (plan to) maintain the same level of consumption at all ages. The life cycle optimizer (i.e. “consumption smoother”) spends more at earlier ages and reduces spending as they age, even if their subjective discount rate (SDR) is equal to the real interest rate. Thus the utility-maximizing retiree is not willing to reduce their initial standard of living simply because of a small probability they will reach age 105. The utility maximizing retiree does not maintain a constant spending rate. They deal with longevity risk by setting aside a financial reserve AND by planning to reduce consumption if that risk materializes in proportion to the survival probability and linked to their risk aversion. All of this, of course, is in the absence of pension annuity income.

To quote Irving Fisher (1930) in his Theory of Interest (page 85): “...*The shortness of life thus tends powerfully to increase the degree of impatience or rate of time preference beyond what it otherwise be...*” and (page 90) “*Everyone at some time in his life doubtless changes his degree of impatience for income...When he gets a little older, if his children are married and have gone into the world he may again have a high degree of impatience for income because he expects to die and he thinks: instead of piling up for the remote future, why shouldn't I enjoy myself during the few years that remain?*”

[3.1] Including Pension Annuities

Going one step further: We now assume that the retiree has access to a Defined Benefit (DB) pension which provides guaranteed lifetime income. The maximum amount of Social Security (S.S.) in the U.S., which is the ultimate real pension annuity, is approximately \$25,000 per individual. So, we examine the behavior of a retiree with 100, 50 and 20 times this amount in their nest egg. In other words, we consider (in the language of the financial planning community) a “high net worth” individual with (i.) \$2,500,000 in investable retirement assets, (ii.) \$1,250,000 in investable retirement assets and (iii.) \$500,000 in investable retirement assets.

Alternatively, one can interpret Table #2 as displaying the optimal policy for 4 different retirees, each with \$1,000,000 in investable retirement assets. The first has no ($\pi=\$0$) pension, the second has a pension of \$10,000 per year ($\pi=\1), the third has a pension of \$20,000 per year ($\pi=\2) and the fourth has a pension of \$50,000 ($\pi=\5).

Table #2: Initial Portfolio Withdrawal Rate (PWR) at Age 65 as a Function of Risk Aversion				
Pension Income	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$
$\pi_0 = \\$0$	6.330%	5.301%	4.605%	4.121%
$\pi_0 = \\$1$	6.798%	5.653%	4.873%	4.324%
$\pi_0 = \\$2$	7.162%	5.924%	5.078%	4.480%
$\pi_0 = \\$5$	8.015%	6.553%	5.551%	4.839%

Note: 5% survival probability to age 100. Equivalent to Gompertz ($m=89.335, b=9.5$). Interest Rate = 2.5%

Table #2 displays the optimal net-withdrawal rates as a function of the above-mentioned risk aversion values. By net-withdrawal rates we mean the amount withdrawn from the actual portfolio. Thus, for example, when the ($\gamma = 4$) retiree has \$1,000,000 in investable assets and a pension of \$50,000, which implies a scaled nest egg of \$100 and a pension ($\pi=\5), the optimal total consumption rate is \$10.551 – of which \$5.00 comes from the pension and \$5.551 is withdrawn from the portfolio. Thus the portfolio withdrawal rate (PWR) is **5.551%**. In contrast if the retiree has the same \$1,000,000 in investable retirement assets and only a \$10,000 pension, then the optimal total consumption rate is \$5.873 at age 65, of which \$1.00 comes from the pension and \$4.873 is withdrawn from the portfolio. Hence, the PWR is **4.85%**. Once again we emphasize that the PWR depends on longevity risk aversion and the level of pre-existing pension income. This is quite clear from equation (#5). The larger the amount of the pre-existing pension income, the greater is the consumption rate (obviously) – and the greater is the PWR, which is not so obvious. Basically, the pension acts as a buffer and allows the retiree to consume more from discretionary wealth. Even at high levels of longevity risk aversion, the chance of living a longer lifespan doesn't "worry" the retiree, as they have a pension to fall back upon should that chance materialize.

Table #2 confirms a number of other results that should now be intuitive. Notice that the optimal spending rate – for a range of pension income and longevity risk aversion levels – is between 8% and 4% when the underlying interest rate is assumed to be 2.5%. In general, adding another 100 basis points to

the investment return assumption adds somewhere between 60 to 80 basis points to the initial PWR. Remember that this is only the initial portfolio withdrawal rate. As time evolves the consumption rate will decline – unless the entire portfolio has been “pensionized” – as displayed in Table #1, for example.

The impact of longevity risk aversion can be described as follows. If the biological law of mortality is described by a Gompertz distribution with modal lifespan value ($m = 89.335$) and dispersion parameter ($b = 9.5$), the longevity risk averse consumer behaves as if the modal value is: ($m^* = m + b \ln \gamma$), but with the same dispersion parameter b . Longevity risk aversion manifests itself by (essentially) assuming you will longer than the biological/medical estimate. It is only Bernoulli ($\gamma = 1$) who behaves as if his modal lifespan is the true modal value. Note that this is not risk-neutrality, which would ignore longevity risk all together and simply focus on planning until the expectation of life.

The closest analogy within the financial world that we can find to our way of approaching the problem is the concept of risk-adjusted investment returns. Basically, a risk-averse investor observes a 10% portfolio return and adjusts it downward based on the volatility of the return and their risk aversion. If the (subjectively) adjusted investment return is under the risk-free rate, the investor shuns the risky asset. Of course this analogy isn't quite correct since the retiree can't shun longevity risk, but the spirit is the same.

Another important takeaway here is the impact of pension annuities on consumption. While the point of this paper is not to advocate or argue for (more) pension annuities and longevity instruments – that is well achieved in the book by Sheshinski (2008) or Brown, Mitchell, Poterba and Warshawsky (2001) – here is yet another way to use equations (#5, #6) to illustrate this point.

Table #3 displays the optimal consumption rate at various ages after retirement, assuming that a fixed percentage of the nest egg is used to purchase a pension annuity at retirement. The cost of each lifetime dollar of income is displayed in equation (#2), which is the expression for the pension annuity factor. So, if 30% of \$100 is “pensionized” the corresponding value of $F_0 = \$70$ and resulting pension annuity income is $\$30/a_{65}^{55}(0.025, 89.335, 9.5) = \1.899 .

Results are at retirement age 65 and planned consumption 15 years later (assuming the individual is still alive) at age 80. We illustrate a variety of different scenarios in which 0%, 20%, 40%, 60% and 100% of

initial wealth of \$100 is annuitized. Note that by “pensionization” we mean the purchase of a non-reversible pension annuity – which is priced by equation (#2) --- based on the going market rate.¹¹

TABLE #3: HOW DOES “PENSIONIZATION” IMPACT RETIREMENT CONSUMPTION?				
Percent of \$100 Pensionized	Retiree with “Medium” Longevity Risk Aversion ($\lambda = 4$)		Retiree with “High” Longevity Risk Aversion ($\gamma = 8$)	
	Consumption at 65	Planned Consumption at 80	Consumption at 65	Planned Consumption at 80
0%	\$4.605	\$4.007	\$4.121	\$3.844
20%	\$5.263	\$4.580	\$4.801	\$4.478
40%	\$5.795	\$5.042	\$5.385	\$5.024
60%	\$6.227	\$5.419	\$5.937	\$5.538
100%	\$6.330	\$6.330	\$6.330	\$6.330

Notes: Assume a fairly priced pension annuity which pays \$1 of lifetime income per \$15.791 premium, under a 2.5% interest rate, based on equation (#2).

Note that Table #3 displays total dollar consumption rates, including the corresponding pension annuity income. These are not the portfolio withdrawal rates which were displayed (in percentages) in Table #2. So, for example, if the (medium risk aversion) retiree uses \$20 of the initial \$100 to purchase a pension annuity that pays \$1.261 for life, then consumption will be $\$1.261 + \$3.997 = \$5.263$ at age 65. The \$3.997 withdrawn from the portfolio of \$80 is an initial portfolio withdrawal rate of 4.997%. In contrast, the retiree with a high degree of longevity risk aversion ($\gamma = 8$) will obtain the same \$1.261 from the \$20 that has been “pensionized” but will only spend \$3.535 from the portfolio (a withdrawal rate of 4.419%), for a total consumption rate of \$4.801 at age 65.

Finally, if the entire nest egg of \$100 is pensionized at age 65, which results in \$6.3303 of lifetime income, then consumption rate is (obviously) constant for life since there is no financial capital from which to draw-down any income. Once again, this illustrates the benefit of converting financial wealth into a pension income flow, since the \$6.3303 of annual consumption is the largest of all consumption

¹¹ This is quite different from the Yaari (1965) tontine annuity, adopted and used in many economic models, in which mortality credits are paid-out instantaneously by adding the mortality hazard rate λ_t to the investment return r . This is why we use the phrase “pensionization” to distinguish from the economist’s use of the term annuitization. The latter assumes a pool in which survivors inherit the assets of the deceased, while the former requires an insurance company or pension fund to guarantee the lifetime payments.

plans. In fact, this is precisely why most financial economists are strong advocates of “pensionizing” of a portion of one’s retirement nest egg.

[3.2] Plotting the Results

Figure #1 displays the optimal consumption path from retirement until death as a function of the retiree’s level of risk aversion (γ in our model). This figure gives yet another perspective on the rational approach to (personal) longevity risk management. Figure #1 uses equation (#5) to trace-out the entire consumption path from retirement at age 65 until age 100.

As we explained earlier, and continue to emphasize, the optimal consumption rate declines with age and in relation to the retiree’s attitude towards financial risk and measured by their coefficient of relative risk aversion (CRRA). The figure plots four cases corresponding to differing levels of CRRA. Notice the consumption rate eventually hits \$5, which is the pension income. So, for example, the CRRA = 2 (i.e. low aversion to longevity risk) consumer will start retirement by withdrawing and consuming 6.55% of their nest egg plus their pension of \$5. The withdrawals from the portfolio will continue and increase until they (rationally) exhaust their wealth at age 95. From the Wealth Depletion Date (WDT) onwards all they consume is their pension¹².

¹² For those readers who are interested in these matters, the consumption function is concave until the WDT, at which point it is non-differentiable and forced equal to the pension annuity income π_0 .

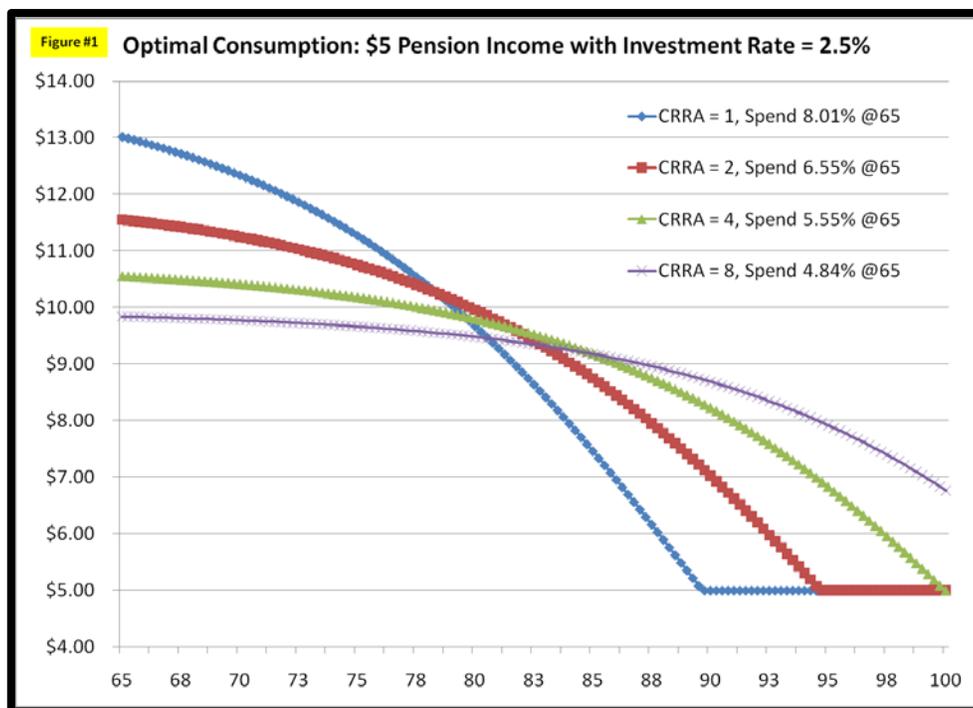
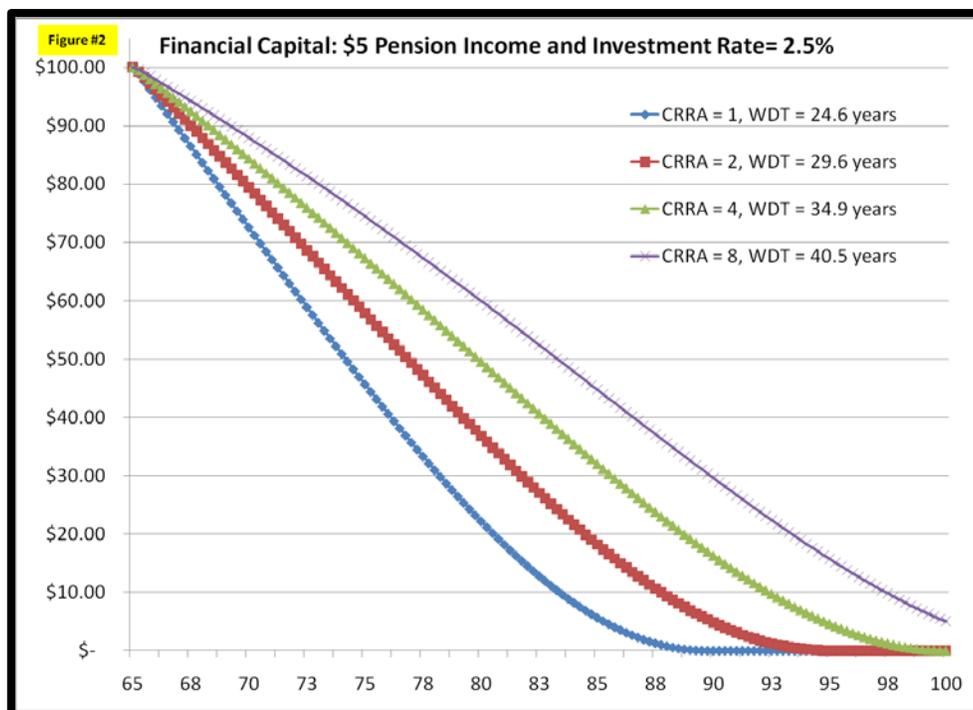


Figure #2 displays the corresponding trajectory for financial capital. At all levels of longevity risk aversion the curve begins at $F_0 = \$100$, and then declines from there. The rate of decline is higher and faster for lower levels of longevity risk aversion. When the coefficient of relative risk aversion (CRRA or γ) is equal to one, the curve declines more rapidly. This is because the individual is “not afraid” of living to an advanced age. They will deplete their wealth after 24.6 years (which is age 90) after which they live on their Social Security Pension annuity (\$5).



In contrast, the retiree with longevity risk aversion $CRRA = 8$ doesn't (plan to) deplete wealth until age 105, and draws-down wealth at a much slower rate compared to the $CRRA = 1$ consumer. Note that when there is no pension annuity income at all, the wealth depletion time is exactly at the end of terminal horizon, which is the last possible age on the mortality table. In other words, wealth is never completely exhausted. This can also be seen from equation (#8) where the only way to get zero (on the right hand side) is when $(\tau p_x) = 0$, which obviously can only happen when $\tau = D$.

[3.4] Reacting to Shocks

Our methodology also allows us to examine the optimal reaction to financial shocks and other unexpected changes to financial capital over the retirement horizon. A good example of this would be someone who experiences a 30% loss in their investment portfolio and is trying to figure-out how to rationally reduce spending to account for the depleted nest egg. The “popular” rule of thumb which suggests that a retiree spend 4% to 5% of their nest-egg adjusted for inflation says nothing about how to update or revise this rule in response to a shock to wealth.

The rational reaction to a “shock” to the portfolio at time s , which results in a portfolio value of $F_s = M$, would be to follow these steps:

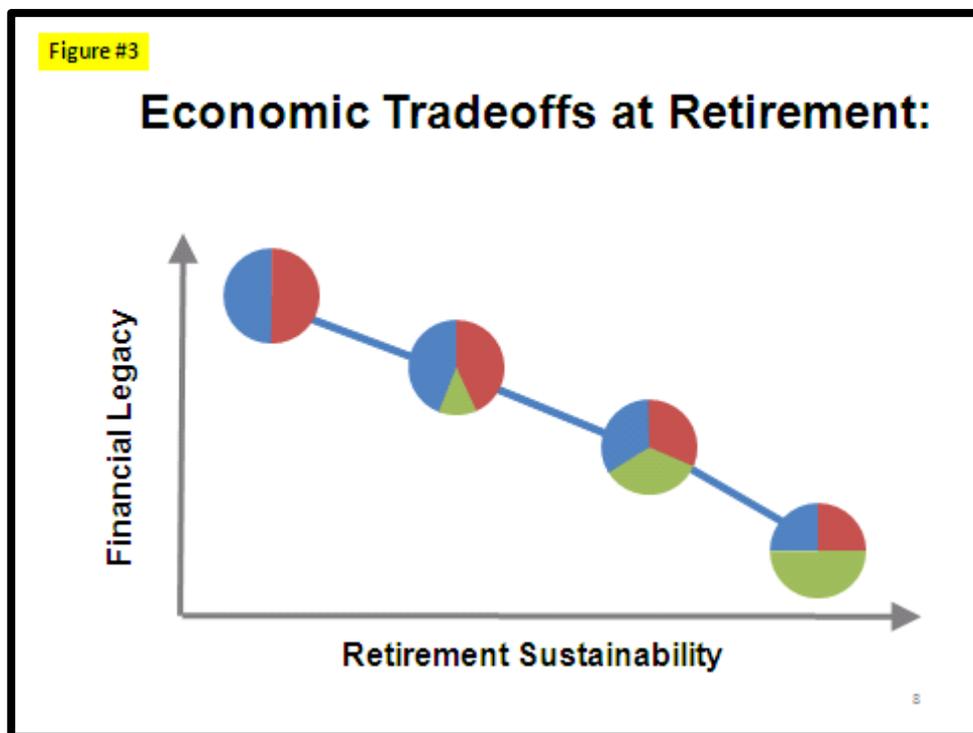
1. Recalibrate the model (again) from time zero, but with $F_0 = M$ and compute the new wealth depletion time from equation (#8).
2. Use equation (#7) to compute the new level of initial consumption c_0^{**} , which is going to be different from (the old) c_s^* because of the shock.
3. Continue retirement consumption from time s onward, based on equation (#5).

For example, let's start with a (CRR=4) retiree with \$100 and with \$2 of pre-existing pension income. The optimal policy is to consume a total of \$67.077 and adjust this downwards over time in proportion to the survival probability to the power of the risk-aversion coefficient. This is dictated by equation (#5.) The conditional on time zero expectation is that at age 70 the financial capital trajectory will be \$86.6687 and consumption will be \$7.077, if the retiree follows the optimal consumption path for the next five years.

Now imagine that the retiree survives five years and experiences some financial (bear market) shock and the portfolio is reduced to only \$60 at age 70, which is 30% less than planned. In this case the optimal plan is to reduce consumption to \$6.113 (by solving the problem from the beginning, but with a starting age of 70), which is a reduction of approximately 14% compared to the original (un-shocked) plan.

Of course, there is a bit of apples and oranges comparison here since (i.) a shock is not really allowed in our deterministic model, and (ii.) the time zero consumption plan is based on a conditional probability of survival as opposed to an unconditional probability of survival. In other words, there is a difference between planning what to do at age 70 if you survive – an event with a probability (${}_5 p_{65} < 1$) -- versus actually surviving to age 70 and then formulating a new plan based on the fact (${}_0 p_{70} = 1$).

Once again, and to conclude this section, all we are saying is that a rational response to an x% drop in one's financial capital is not to reduce consumption by the same x%. This is a corollary of lifecycle smoothing and is a direct result of our model as well.



[4.] Summary and Conclusion:

To a financial economist, the optimal retirement consumption (withdrawal) rate, asset allocation (investments) and product allocation (insurance) is a complicated function of mortality expectations, economic forecasts and the tradeoff between the preference for retirement sustainability versus the desire to leave a financial legacy (bequest motives). The tradeoff is illustrated in Figure #3. In theory a retiree can spend more if they are willing to leave less of a financial legacy, and should spend less if they desire a large legacy. Optimization of investments and insurance products take place on this retirement frontier. Ergo, a simple rule that advises all retirees to spend $x\%$ of their nest egg adjusted up or down in some ad hoc manner, is akin to the broken clock which tells time correctly only twice a day.

Naturally, we are not the first authors and certainly will not be the last to criticize the “spend 4%” approach to retirement income planning. For example, Bill Sharpe and co-authors (2009) recently wrote:

“...The 4% rule and its variants finance a constant, non-volatile spending plan using a risky, volatile investment strategy. Two of the rule’s inefficiencies—the price paid for funding its unspent surpluses and the overpayments for its spending distribution—apply to all retirees, independent of their preferences....”

We concur with this assessment, but our focus in this paper is to actually illustrate what a lifecycle model does say about optimal consumption rates. Our intention was to contrast *ad hoc* recommendations with “advice” that a financial economist would give to a utility-maximizing consumer (Mr. Spock on Planet Vulcan), and see if there is any overlap and by how much, exactly, it differs.

In particular our intention was to shine light on the aversion to longevity risk – the uncertainty of human lifespan – and to examine precisely how this impacts the optimal spending rate.

Computationally we solved a basic life-cycle model (LCM), but one that was realistically calibrated to actuarial mortality rates via the Gompertz law of mortality. Quite fortuitously and using techniques from the Calculus of Variations we were able to obtain closed-form expressions for the optimal consumption rate and the net portfolio withdrawal rate (NWR) as well as trajectory of financial capital during retirement in terms of the Gamma function.

Our main qualitative insights are as follows:

1. The optimal initial portfolio withdrawal rate (PWR) which the so-called “planning literature” has claimed should be an exogenous percentage of one’s retirement nest egg actually depends quite critically on both the consumer’s risk aversion – where risk is longevity and not just market returns – as well as any pre-existing pension annuity income. For example, if we assume that the ongoing real (after-inflation) investment return of a portfolio is 2.5% per annum, then for individuals who are highly risk-averse the optimal initial PWR can be as low as 3%, and for individuals who are less risk-averse it can be as high as 7%. The same applies to the existence of pension annuity income. The greater the amount of pre-existing pensions (for example Social Security) the greater the initial PWR, all else being equal. Of course, if one assumes a healthier retiree and/or lower inflation-adjusted returns the optimal initial PWR is lower as well.
2. The optimal consumption rate (c_t^*), which is the total amount of money consumed by the retiree in any given year including all pension income, is a declining function of age. In other words, retirees (on Planet Vulcan) should consume less at older ages than younger ages. The consumption rate is proportional to the survival probability (${}_t p_x$) and is a function of risk aversion, even when the subjective rate of time preferences (ρ) is equal to the interest rate. In other words it simply does not make any sense to target a fixed constant standard of living or constant portfolio withdrawal rate. The rational consumer – planning at age 65 -- is willing to sacrifice some income at the age of 100 in exchange for more income at the age of 80. Stated

differently, giving the age of 100 the same preference weight as the age of 80 can only be explained within a lifecycle model if the subjective discount rate (ρ_t) is a time-dependent function that exactly offsets the declining Gompertz survival probability.

3. The interaction between (longevity) risk aversion and survival probability is quite important. In particular, the impact of risk aversion is to increase the effective probability of survival. In other words, imagine two retirees with the same amount of initial retirement wealth and pension income (and the same subjective discount rate) but with differing levels of risk aversion (γ). The individual with greater risk aversion behaves as if their life expectancy is higher. In particular they behave as if it is increased by an amount proportional to $\ln[1/\gamma]$.
4. The optimal trajectory of financial capital declines with age. Moreover, for individuals with pre-existing pension income it is rational to spend-down wealth by some advanced age and live exclusively on the pension income (for example Social Security). The wealth depletion time (WDT) can be at age 90 – or even age 80 when the pension income is sufficiently large. Greater (longevity) risk aversion which is associated with lower consumption induces greater financial capital at all ages. Either way, there are bag ladies on Planet Vulcan.
5. The rational reaction to portfolio shocks is non-linear and dependent on when the shock is experienced as well as the amount of pre-existing income. In other words one does not reduce their portfolio withdrawal rate (PWR) by the exact amount of a financial shock unless their risk aversion is ($\gamma = 1$) à la Bernoulli. So, if the portfolio suffers an unexpected loss of 30%, the retiree does not consume 30% less as a result of the shock. Thus, even if one was to arbitrarily adopt a constant spending rate policy -- say 5% of the initial portfolio value -- one still needs guidance on how to react to shocks.
6. Pensions are quite valuable from a number of perspectives and converting some of the initial nest egg into a stream of lifetime income increases consumption at all ages regardless of the exact cost of the pension annuity. In other words even when interest rates are low and the cost of \$1 of lifetime income is high the net effect is that it increases consumption. Note that we are careful to distinguish between real world pension annuities -- in which the buyer hands-over an irreversible sum in exchange for a constant real stream – and Tontine annuities which are at the foundation of most economic models but are completely unavailable.
7. Our final result that we have not emphasized within the paper is counter-intuitive and perhaps even controversial. To wit: borrowing against pension income might be optimal at advanced ages. For individuals with relatively large pre-existing (Defined Benefit) pension income, it might

make sense to pre-consume (and enjoy) the pension while they are still young. The lower the (longevity) risk aversion, the more optimal this becomes.

Of course, the “cost” of having a simple analytic expression – described by equations (#1) to equation (#8) -- for the quantities of interest, is that we had to assume a deterministic investment return. In a sense although we used a safe and conservative assumed return for most of the displayed numerical examples, we have essentially ignored the last 50 years of portfolio modeling theory (although perhaps that is a good thing!). Recall however that our attempt was to shed light on the often-quoted rules of thumb and how they relate to longevity risk, as opposed to developing a full-scale stochastic optimization model for retirement.

How might a full stochastic model – with possible shocks to health and their related expenses as well -- change our simple consumption policies? Well, assuming one could agree on a reasonable model and parameters for long-term portfolio returns, the risk-averse retiree would be exposed to the risk of a negative (early) shock and would plan for this in advance by consuming less. However, with a full menu of investment assets and products available, the retiree would be free to optimize around pension annuities and other downside-protected products, in addition to long-term care (LTC) insurance and other retirement products. In other words, even the formulation of the problem itself becomes much more complex.

More importantly, the optimal allocation depends on the retiree’s preference for personal consumption versus bequest, as illustrated in Figure #3. A product and asset allocation that is suitable (maximizes utility) for a consumer with no bequest or legacy motives – those in the lower left-hand corner of the figure – is quite different from the optimal portfolio for someone with strong legacy preferences. This paper assumed that the retiree’s objective is to maximize utility (satisfaction, happiness) of lifetime consumption without any consideration for the value of bequest or legacy. This was quite clear from equation (#1).

That said, we are currently working on a sequel to this paper in which we derive the optimal portfolio withdrawal rate (PWR) – with pension and tontine annuities -- in a more robust capital markets environment *a la* Richards (1975) and Merton (1971).

One thing seems clear: risk-aversion and pension annuities are (still) very important when giving advice regarding optimal portfolio withdrawal rates. That was the main message of this paper and it does not change on Planet Earth.

Bibliography and References

Arnott, R.D. (2004), Editor's Corner: Sustainable Spending in a Lower Return World, *Financial Analysts Journal*, September/October 2004, pg 6-9.

Babbel, D.F. and C.B. Merrill (2006), Rational Decumulation, working paper *Wharton Financial Institutions Centre*.

Bengen, W.P. (1994), Determining withdrawal rates using historical data, *Journal of Financial Planning*, October 1994, Vol. 7(4), pg. 171-181.

Bodie, Z. and J. Treussard (2007), Making investment choices as simple as possible but not simpler, *Financial Analysts Journal*, Vol. 63(3), pg. 42-47

Bodie, Z., D. McLeavey and L.B. Siegel (editors, 2008), *The Future of Life-Cycle Saving and Investing* (second edition), published by the Research Foundation of the CFA Institute.

Bodie, Z., J.B. Detemple, S. Ortuba and S. Walter (2004), Optimal consumption-portfolio choice and retirement planning, *Journal of Economic Dynamics and Control*, Vol. 28, pg. 1115-1148.

Brown, J.R., O.S. Mitchell, J.M. Poterba and M.J. Warshawsky (2001), *The Role of Annuity Markets in Financing Retirements*, MIT Press, Cambridge.

Butler, M. (2001), Neoclassical life-cycle consumption: a textbook example, *Economic Theory*, Vol. 17, pg. 209-221.

Chen, P., R.G. Ibbotson, M.A. Milevsky and K.X. Zhu (2006), Human Capital, Asset Allocation and Life Insurance, *Financial Analysts Journal*, Vol. 62(1), pg. 97-109.

Cooley, P.L., C.M. Hubbard, and D.T. Walz, (1998) Retirement Spending: Choosing a Withdrawal Rate That Is Sustainable. *Journal of the American Association of Individual Investors*, Vol. 20(1), pg. 39-47.

Deaton, A. (1991), Saving and liquidity constraints, *Econometrica*, Vol. 59(5), pg. 1221-1248

Fisher, I. (1930), *The Theory of Interest: As Determined by Impatience to Spend Income and Opportunity to Invest It*, The Macmillan company, New York.

Hanna, S., J.X. Fan and Y.R. Chang (1995), Optimal lifecycle savings, *Financial Counseling and Planning*, Vol. 6, pg. 1-14.

Huang, H. and M.A. Milevsky (2010), Calibrating Yaari to 21st Century Products: Tontines, Annuities and Optimal Retirement Consumption, *Working Paper, York University and the IFID Centre*, www.ifid.ca

Kotlikoff, L. (2008), Economics' Approach to Financial Planning, Boston University working paper and *Journal of Financial Planning*, March.

Liu, Hong (2005), Lifetime consumption and investment: Retirement and Constrained Borrowing, *working paper, John M. Olin School of Business*.

Leung, S. F. (1994), Uncertain Lifetime, The Theory of the Consumer and the Life Cycle Hypothesis, *Econometrica*, Vol. 62(5), pg. 1233-1239

Leung, S. F. (2007), The existence, uniqueness and optimality of the terminal wealth depletion time in life-cycle models of saving under certain lifetime and borrowing constraint, *Journal of Economic Theory*, Vol. 134, pg. 470-493.

Milevsky, M.A. (2006), *The Calculus of Retirement Income: Financial Models for Pensions and Insurance*, Cambridge University Press, UK

Martin-Jimenez, S. and A. R. Sanchez Martin (2007), An evaluation of the life cycle effects of minimum pensions on retirement behavior, *Journal of Applied Econometrics*, Vol. 22, pg. 923-950.

Merton, R.C. (1971), Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, Vol. 3(4), pg. 373-413.

Modigliani, F. and R. Brumberg (1954), Utility analysis and the consumption function: An interpretation of cross-section data, published in *Post Keynesian Economics*, New Brunswick, Rutgers University Press.

Modigliani, F. (1986), Lifecycle, individual thrift and the wealth of nations, *American Economic Review* Vol. 76(3), pg. 297-313.

Phelps, E.S. (1962), The accumulation of risk capital: a sequential utility analysis, *Econometrica*, Vol. 30(4), pg. 729-743.

Ramsey, F.P. (1928), A mathematical theory of saving, *The Economic Journal*, Vol. 38(152), pg.543-559

Richard, S.F. (1975), Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model, *Journal of Financial Economics*, Vol. 2, pg. 187-203.

Samuelson, P.A. (1969), Lifetime portfolio selection by dynamic stochastic programming, *The Review of Economics and Statistics*, Vol. 51, pg. 239-246.

Sharpe, W. F., Scott, J. S. and Watson, J.G. (2007), Efficient Retirement Financial Strategies (July 2007). Pension Research Council Working Paper Series. Available at SSRN: <http://ssrn.com/abstract=1005652>

Scott, J.S., Sharpe, W.F. and J. G. Watson (2009), The 4% Rule—At What Price?, *Journal of Investment Management*, Third Quarter, www.joim.com

Scott, J.S. (2008), The longevity annuity: An annuity for Everyone?, *Financial Analysts Journal*, Vol. 64, pg. 40-48.

Sheshinski, E. (2008), *The Economic Theory of Annuities*, Princeton University Press, New Jersey

Thaler, R.H. and C.R. Sunstein, (2008) *Nudge: Improving Decisions About Health, Wealth and Happiness*, Yale University Press, 2008.

Yaari, M.E. (1965), Uncertain Lifetime, Life Insurance and the Theory of the Consumer, *The Review of Economic Studies*, Vol. 32(2), pg. 137-150