



# Monte Carlo Simulation of Models in Actuarial Finance

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## Agenda for Today

- Explain theoretical basis for Monte Carlo Simulations (MCS) and distinguish from Futures Probability Simulations (FPS).
- Introduce the Brownian Motion which is the stochastic process underlying most financial and actuarial **diffusion** models.
- Demonstrate how to efficiently simulate any diffusion process in Excel.
- Case study illustration: Valuation of Guaranteed Annuity Options (GAO)

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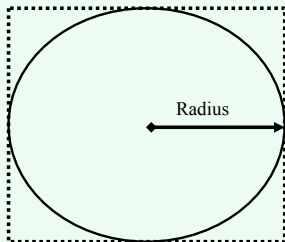
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## Example: Computation of $\pi$

Area =  
 $R * R * \pi$



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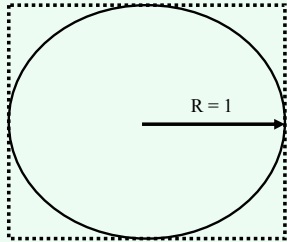
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### Question: What is the Ratio of the Area of the Circle vs. Square?

Area of Circle is  $\pi$   
 Area of Square is 4

$$\frac{3.14159}{4} \approx 78.53\%$$




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### Famous Problem in Mathematics

- Question: What is the exact value of  $\pi$ ?
- Scholars spent years trying to compute.
- Trick: Use Monte Carlo Simulation
- The ratio of the probabilities approximates the ratio of the areas....which therefore approximates the value of  $\pi$ .

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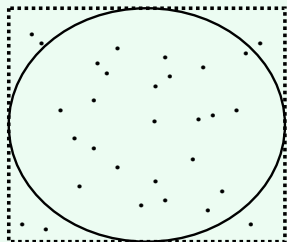
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### Throwing Darts at a Board...

Area of Circle is  $\pi$   
 Area of Square is 4

$$\frac{\# \text{ Inside}}{\# \text{ Total}} = \frac{\pi}{4}$$

$$4 \left( \frac{24}{30} \right) = 3.2 \approx 3.14159$$




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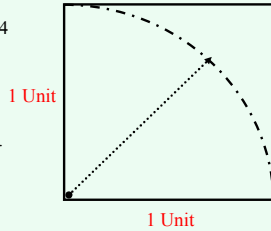
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## Equally Efficient: Quarter Square

Ratio of In-to-Out is still:  $\pi/4$

Generate two independent  $[0,1]$  random variables:  $X, Y$ .

$$X^2 + Y^2 \leq 1$$




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## Formal Simulation:

$$x \equiv U[0,1], y \equiv U[0,1]$$

$$f(x, y) = \begin{cases} 4, & \text{if } x^2 + y^2 \leq 1 \\ 0, & \text{if } x^2 + y^2 > 1 \end{cases}$$

$$E[f(x, y)] = \pi$$

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## Number of Trials and Accuracy

- Mean.... $m$
- Number of Trials.... $n$
- Standard Deviation.... $s$
- Standard Error of Estimates... $\frac{s}{\sqrt{n}}$
- The  $\alpha\%$  Confidence Interval for True Value

$$m - \frac{\Phi(\alpha)s}{\sqrt{n}} \leq V \leq m + \frac{\Phi(\alpha)s}{\sqrt{n}}$$

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## Simulation in Excel: Tools / Data Analysis / Random

Average (n=100): 3.2000  
 Standard Deviation: 1.6081  
 Standard Error of Estimate: 0.1608

0.0013	0.5636	4
0.1933	0.8087	4
0.5850	0.4799	4
0.3503	0.8960	4
0.8228	0.7466	0
0.1741	0.8589	4
0.7105	0.5135	4
0.3040	0.0150	4
0.0914	0.3645	4
0.1473	0.1659	4
0.9885	0.4457	0
0.1191	0.0047	4



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## More Simulations..... .....Greater Accuracy

Average (n=1000): 3.1120  
 Standard Deviation: 1.6632  
 Standard Error of Estimate: 0.0526

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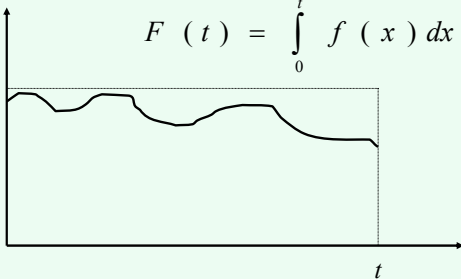
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## Monte Carlo Simulation: Evaluation of Complex Integrals

$$F(t) = \int_0^t f(x) dx$$



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## Improving Simulation: Efficiency Techniques

- Antithetic Variables:
  - *Exploit Symmetry*
- Control Variate:
  - *Exploit Analytics*
- Importance Sampling:
  - *Exploit Relevance*
- Stratified Sampling:
  - *Exploit Regions*

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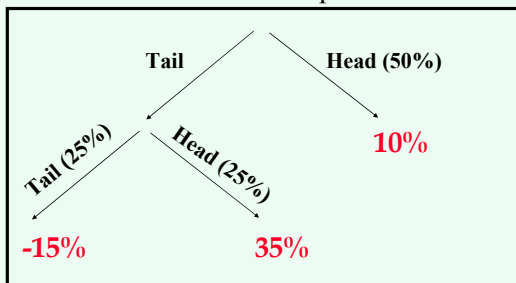
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## Future Probability Simulation (FPS): The Coin Toss Experiment




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## Long-term expectations.

- The **Arithmetic Mean** return from this investment is exactly **10%**.
- Imagine you start with \$100 and stay invested for twenty years. What is your expected wealth at the end of the period?
- Answer  $100 \times (1.1)^{20} = \$672$ , is **wrong!**
- True number is  $100 \times (1.0855)^{20} = \$516$

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## Arithmetic vs. Geometric Mean

- The **Arithmetic Mean** of the coin-toss gamble is **10%**; the **Geometric Mean** of the coin-toss gamble is only **8.55%**
- The gap is often called the 'volatility drag'.
- The **GM** is a better indication of growth over the long term, compared to **AM**.
- Why? Imagine you earn **-15%** in 1<sup>st</sup> year, and **35%** in the 2<sup>nd</sup> year. The total return is **14.75%**, versus **21%** from earning the Arithmetic Mean of **10%** each year.

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Greater volatility reduces the **GM** and the growth rate, even if **AM** remains unchanged.

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	<b>GM</b>
10%	35%	-15%	<b>8.55%</b>
10%	40%	-20%	<b>7.89%</b>
10%	45%	-25%	<b>7.10%</b>
10%	50%	-30%	<b>6.17%</b>
10%	60%	-40%	<b>3.82%</b>

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## Simulation Fact and Fiction

- Important to distinguish between Monte Carlo Simulations (MCS) and Future Probability Simulations (FPS).
- Use only when analytics are not available.
- It is impossible to generate a *truly* random sequence of numbers, unless you use a quantum physics generator.
- Note the difference between quasi-random (low discrepancy) and pseudo-random.

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## The Brownian Motion

- 1 Continuous process that starts at zero.
- 2 Stationary and independent increments.
- 3 Mean of zero, variance of time.
- 4 Normal distribution at any point in time.

$$B_t \equiv N(0, \sqrt{t})$$

$$dB_t \approx \pm \sqrt{dt}$$

$$\Delta B_t \approx N(0, \sqrt{\Delta t})$$

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Sample Path of Standard Brownian Motion:  
 $\Delta t = 1/365, N = 500$




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Sample Path of Standard Brownian Motion:  
 $\Delta t = 1/365, N = 500$




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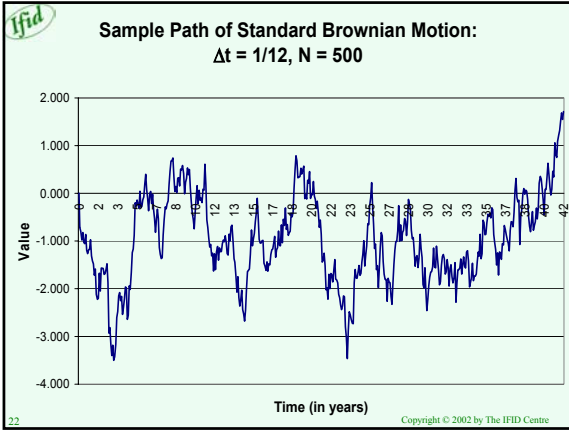
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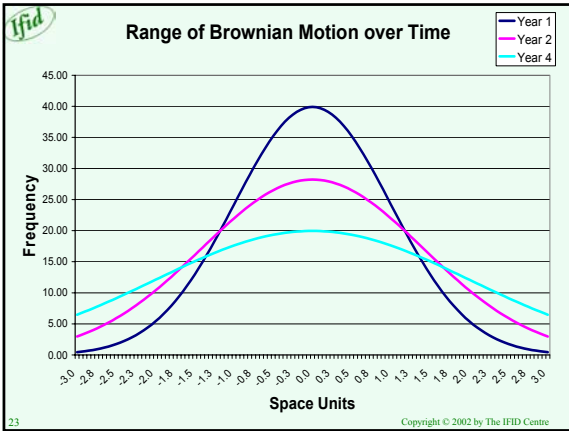
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**IFID**

$$B_{i+1} = B_i + Z \sqrt{\Delta t}$$

Period #	Time (years)	N(0,1)	BM.t
0	0	-1.15449	0.00000
1	0.083	-2.48376	-0.71700
2	0.167	-0.28076	-0.79805
3	0.250	-0.62106	-0.97733
4	0.333	0.53488	-0.82292
5	0.417	-0.72082	-1.03101
6	0.500	-0.08498	-1.05554
7	0.583	0.66804	-0.86269
8	0.667	-1.15628	-1.19648
9	0.750	-0.23780	-1.26513
10	0.833	0.27591	-1.18548
11	0.917	0.04090	-1.17367
12	1.000	0.69341	-0.97350

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## Facts about Brownian Motion

$$E[B_t | B_s] = B_s \quad t \geq s$$

$$\Pr[B_t \leq C | B_s = k] = N\left(\frac{C - k}{\sqrt{t - s}}\right)$$

$$\lim_{t \rightarrow \infty} \frac{B_t}{t} = 0$$

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## Correlated Brownian Motions

$$\begin{matrix} B_t^1 \\ B_t^2 \end{matrix} \longrightarrow \text{Move together}$$

$$\text{cor}(dB_t^1, dB_t^2) = \rho$$

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## Geometric Brownian Motion

$$S_t = S_0 \exp\{\alpha t + \sigma B_t\}$$

Exponential Growth with Proportional White Noise

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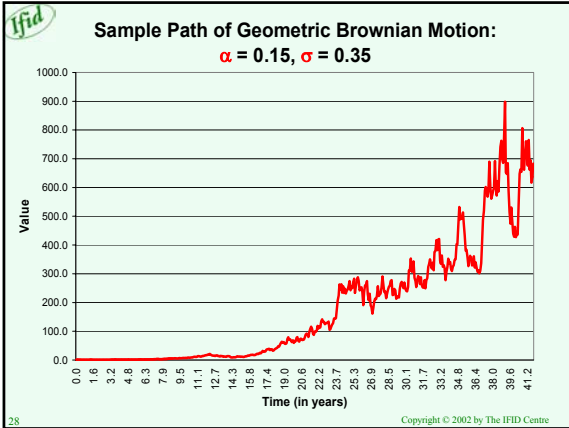
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Simulation of GBM in Excel

Growth Coefficient (alpha): 0.150  
 Diffusion Coefficient (sigma): 0.350

Period #	Time (years)	N(0,1)	BM.t	(a)+s(Bm.t)	EXP((a)+s(Bm.t))
0	0	0.0000	0.00000	0.00000	1.000000
1	0.083	-1.6725	-0.48281	-0.15648	0.855145
2	0.167	1.1240	-0.15833	-0.03041	0.970043
3	0.250	-1.8332	-0.68752	-0.20313	0.816170
4	0.333	0.6702	-0.49404	-0.12291	0.884340
5	0.417	-0.8979	-0.75325	-0.20114	0.817799
6	0.500	-1.9852	-1.32634	-0.38922	0.677587
7	0.583	1.2263	-0.97234	-0.25282	0.776607
8	0.667	-0.9542	-1.24780	-0.33673	0.714101
9	0.750	-1.2486	-1.60825	-0.45039	0.637381
10	0.833	-0.7414	-1.82228	-0.51280	0.598819
11	0.917	0.8005	-1.59119	-0.41942	0.657431
12	1.000	0.4055	-1.47414	-0.36595	0.693538

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Alternative Representation:

$$S_t = S_0 \exp \{ \alpha t + \sigma B_t \}$$

$$\Updownarrow$$

$$dS_t = (\alpha + \sigma^2 / 2) S_t dt + \sigma S_t dB_t$$

Stochastic Differential Equation

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### General Stochastic Differential Equation a.k.a. Diffusion Process

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dB_t$$

Drift                  Diffusion  
Term                  Term

Think:  $\Delta S_{t+1} = \mu_t S_t \Delta t + \sigma_t S_t \Delta B_{t+1}$

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### Vasicek Process (interest rates)

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dB_t$$

Speed of Reversion

Volatility

Long-Term Rate

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### The CIR (1985) Process:

$$dr_t = \kappa(m - r_t)dt + \sigma\sqrt{r_t}dB_t$$

a.k.a. the mean-reverting square-root process

This model is quite popular because of analytic tractability  
(even though it does not fit the data very well...)

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### CIR: Steady State Distribution

$$r_T \approx \text{Gamma}$$

$$E [ r_T ] = m$$

$$SD [ r_T ] = \sigma \sqrt{\frac{m}{2 \kappa}}$$

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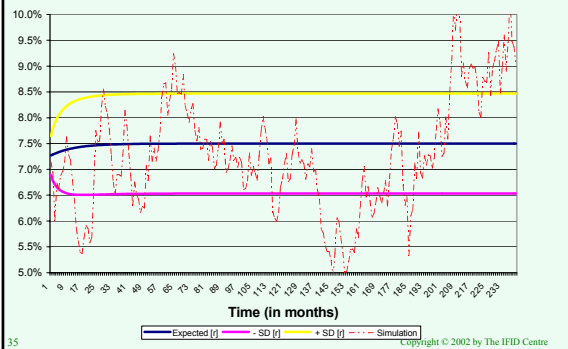
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### Sample Path and Range for Interest Rate (CIR 85):

Assuming:  $r_0 = 7.25\%$ ,  $m = 7.50\%$ ,  $\kappa = 1.0$ ,  $\sigma = 5\%$




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### Sample Path and Range for Interest Rate (CIR 85):

Assuming:  $r_0 = 7.25\%$ ,  $m = 7.50\%$ ,  $\kappa = 0$ ,  $\sigma = 5\%$




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### Simulating the CIR Diffusion: Monthly Approximation

$r_i$	$\Delta B_{i+1}$	$\Delta r_{i+1}$
$r_0$	$N(0, \sqrt{1/12})$	$k(m-r_0)/12 + \sigma\sqrt{r_0}\Delta B_1$
$r_1 = r_0 + \Delta r_1$	$N(0, \sqrt{1/12})$	$k(m-r_1)/12 + \sigma\sqrt{r_1}\Delta B_2$
$r_2 = r_1 + \Delta r_2$	$N(0, \sqrt{1/12})$	$k(m-r_2)/12 + \sigma\sqrt{r_2}\Delta B_3$
$r_3 = r_2 + \Delta r_3$	$N(0, \sqrt{1/12})$	$k(m-r_3)/12 + \sigma\sqrt{r_3}\Delta B_4$




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### Why not Random Hazard Rates?

- We can apply the same kind of thinking to the evolution of the **Force of Mortality** curve.
- We are unsure of mortality patterns in the future, similar to interest rates and other economic variables.
- The **FoM** acts like a forward curve which changes over time.
- The greater the volatility, the higher the value of any embedded option.
- Hedge life annuities (short longevity risk) with life insurance (long longevity risk)

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### An **Actuarial Finance** Model for Mortality Contingent Claims

$$({}_t p_x) = E^Q[\exp\{-\int_0^t h_{x+s} ds\}]$$

$$\lambda_{t+x} = \frac{d \ln[{}_t p_x]}{dt}$$

$$\bar{a}_x = \int_0^\infty E^Q[\exp\{-\int_0^t (r_s + h_{x+s}) ds\}] dt$$

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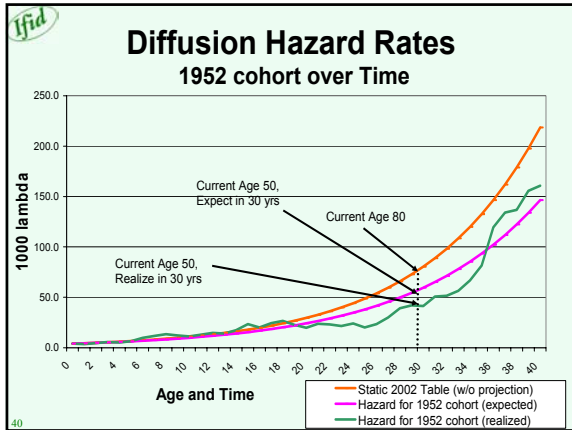
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A Hazard Rate Process:

$$h_t = h_0 \exp\{gt + \xi Y_t\}$$

$$dY_t = -\kappa Y_t + dB_t$$

Mean Reverting Brownian-driven Gompertz (MaRBoG) model.

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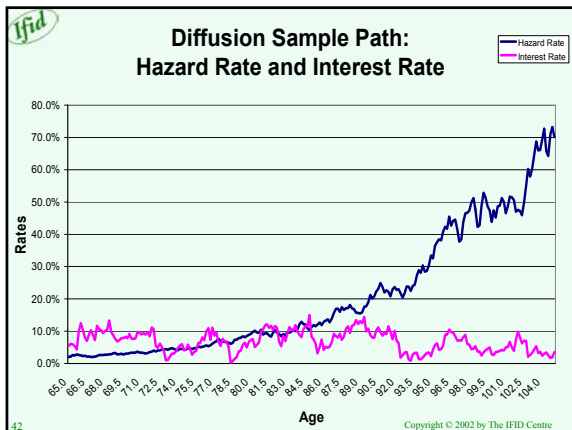
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### Simulation Case Study

- A 45 year-old is given the **guaranteed option** -- at a 180 to 1 factor -- to convert their accumulated investment savings into a monthly immediate life annuity, at age 65.
- How much to **charge** for this option?
- What is the proper **reserve** for this option?
- Main Question: What is Random?
  - Annuity Prices (Yield Curve) in 20 Years.
  - Investment Returns During the Next 20 Years
  - Lapsation and Adoption Rates.

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### Traditional Annuity Pricing:

$$a_x = (1+l) \sum_{i=1}^{\infty} \frac{({}_iP_x)(c_i)}{(1+r_i)^i}$$

Annuity Factor

- Three Ingredients:
- 1.) Explicit Loading
  - 2.) Mortality Rates
  - 3.) Discount Rates

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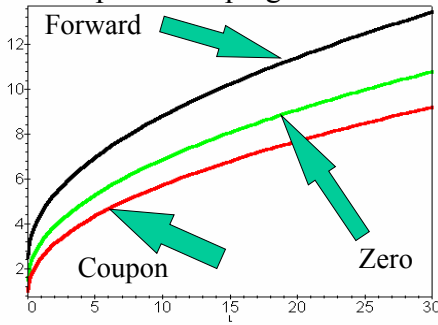
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### Upward Sloping Curve




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## Annuity Simulation

- Focus on the **probability distribution** of immediate annuity factors in 20 years.
- Simulate the **yield curve** assuming the CIR 1985 process for interest rates.
- Simulate future **hazard rates** by projecting mortality and incorporating uncertainty.
- This is a **No Arbitrage** approach. (Note the difference between **Q** and **P** measures.)
- Value-at-Risk, Quantile Reserves, etc.

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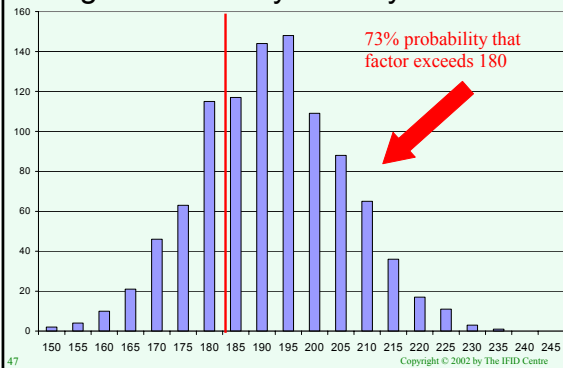
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## Age 65: Monthly Annuity Factor:




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## Final Words...

- A **Monte Carlo Simulation** (MCS) is an efficient method for computing (integral) values, when analytic techniques are not available.
- **Future Probability Simulations** (FPS) are based on the notion that historical frequencies repeat themselves.
- Results are sensitive to the model used.
- **500 runs are not enough!**

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## More Information...

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- J. Cox, J. Ingersoll, S. Ross, (1985) "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol. 53.
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