Financial and Actuarial Analysis of Retirement Savings Products

Dr. Moshe A. Milevsky
Finance Professor
York University
Toronto, Canada

Lecture #2: Agenda

- What is something WORTH?
- Actuarial vs. Financial Economic view.
- Theoretical Pricing of Contingent-Claims in a Complete Binomial Market.
- Product Case Study: Investment Savings with Guaranteed Minimum Death Benefit. (a.k.a. unit-linked contract)
- Main Conclusions...

Theoretical Question #1: How Much to Charge?

- Age 100 pure endowment policy pays $20 in one year, conditional on survival.
- Probability of survival (IAM2000) is 75%.
- Opportunity cost of funds is 10%.
- Fair actuarial premium (no loading) is:

$$\$13.636 = \frac{(0.75)(\$20)}{1.10}$$

Theoretical Question #2: How Much to Charge?

- You must pay \$20, in one year, if the temperature in Buenos Aires exceeds 45c, during the year. You pay nothing otherwise.
- Meteorologist estimate the probability of this event is 75%.
- Opportunity cost of funds is 10%.
- Fair insurance premium (no loading) is:

$$\$13.636 = \frac{(0.75)(\$20)}{1.10}$$

Theoretical Question #3: How Much to Charge?

- You are obligated to pay \$20, in exactly one year, if the Dow Jones Industrial Average exceeds 11,000 by the end of the year.
- Stock market experts estimate the probability of event is 75%.
- Cost of funds is 10%; fair premium is?

$$\$13.636 = \frac{(0.75)(\$20)}{1.10}$$

Theoretical Question #3: How Much to Charge?

- You are obligated to pay \$20, in exactly one year, if the Dow Jones Industrial Average exceeds 11,000 by the end of the year.
- Stock market experts estimate the probability of event is 75%.
- Cost of funds is 10%; fair premium is?

$$\$13.636 = \frac{(0.75)(\$20)}{1.10}$$

- WRONG! Why?

Pricing Theory: In Complete Markets

Assume you observe the market prices:

$$\begin{pmatrix} \text{today..} \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \text{tomorrow..} & \\ \text{good} & \text{bad} \\ 2 & 2 \\ 3 & 0.5 \end{pmatrix}$$

These securities can be purchased, or sold (short), without any restrictions.

An investment bank would like to manufacture a new security that only pays $1 in the good state, and nothing in the bad state. What is it worth?

$$\begin{pmatrix} \text{today..} \\ x \end{pmatrix} \Rightarrow \begin{pmatrix} \text{tomorrow..} & \\ \text{good} & \text{bad} \\ 1 & 0 \end{pmatrix}$$

Initial Thinking...

- If I believe the good state will occur, I will pay more for gamble (security).
- If I believe the bad state will occur, I will pay less for the gamble (security).
- Perhaps I should look at historical probabilities of good and bad states.
- Perhaps it depends on personal utility and attitude towards economic risk?

Interesting Fact:

- We can manufacture (create, hedge, synthesize) this security with the help of simple linear algebra.
- This replication scheme will help us find a price for the contingent-claim.
- Everybody must agree on the price and it is invariant to preferences.

Notice: We can hold B units of the first security (bond) and Δ units of the second security (stock) so that:

$$\begin{pmatrix} \text{Good} & \text{Bad} \\ 3\Delta + 2B & 0.5\,\Delta + 2B \\ = 1 & = 0 \end{pmatrix}$$

This leads to a portfolio of Δ = +0.4, and B = -0.1 units to manufacture the redundant contingent claim.

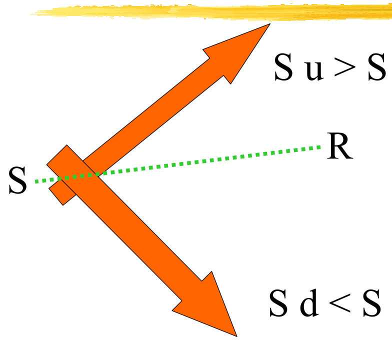
No Arbitrage (NA) Pricing:

- By the method of replication, or manufacturing, or synthesizing, the fair price of the security must be:

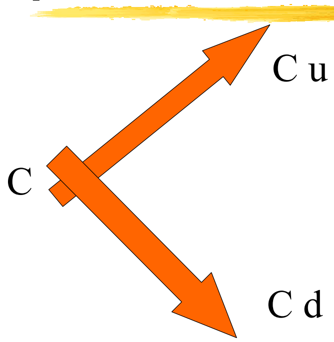
$$x = (+0.4)(1) + (-0.1)(1) = 0.3$$

- Notice that we have not mentioned the words probability, utility or even risk!

Same Model…More Notation



Option Notation



Question: How much would you pay for a one-period call option?

Answer: Solve the system of equations.

$$Su \ \Delta + RB = c_u$$
$$Sd \ \Delta + RB = c_d$$

This implies that:

$$\Delta = \frac{c_u - c_d}{(u-d)S}, \quad B = \frac{uc_d - dc_u}{(u-d)R},$$

which then leads to:

Binomial Option Pricing Formula (BOPF)

$$c_0 = S\Delta + B = \frac{\left[\frac{R-d}{u-d}\right]c_u + \left[\frac{u-R}{u-d}\right]c_d}{R}$$

Numerical Example:

$$S=100, k=200, R=2, u=3, d=0.5$$
$$\Delta=0.4, B=-10, c_0=30$$

Why is this Arbitrage? (+)

Imagine the investment banks sells the security for x=0.35>0.30 (by mistake?). In this case, you would sell (short) the security for $0.35, and buy 0.4 units of the underlying stock, financed by selling (short) 0.1 units of the bond.

$$\begin{pmatrix} \text{today...} & \\ \text{Option} & +0.35 \\ \text{Stock} & -0.40 \\ \text{Bond} & +0.10 \\ \text{Cash Flow} & +0.05 \end{pmatrix} \Rightarrow \begin{pmatrix} \text{tomorrow...} & \\ -1 & 0 \\ +0.4(3) & +0.4(0.5) \\ -0.1(2) & -0.1(2) \\ 0 & 0 \end{pmatrix}$$

This guarantees a risk-free return of $0.05, which can be scaled up to infinity.

Why is this Arbitrage (-)

Imagine the investment banks sells the security for x = \$0.22 < \$0.3 (by mistake?). In this case, you would buy (long) the security for \$0.22, and sell (short) 0.4 units of the underlying stock, investing the difference in 0.1 units of the bond.

$$\begin{pmatrix} \text{today...} & \\ \text{Option} & -0.22 \\ \text{Stock} & +0.40 \\ \text{Bond} & -0.10 \\ \text{Cash Flow} & +0.08 \end{pmatrix} \Rightarrow \begin{pmatrix} \text{tomorrow...} & \\ +1 & 0 \\ -0.4(3) & -0.4(0.5) \\ +0.1(2) & +0.1(2) \\ 0 & 0 \end{pmatrix}$$

This guarantees a risk-free return of \$0.08, which can be scaled up to infinity (in theory).

Assumptions for Manufacturers

- The payoff structure is known in advance.
- The market is complete.
- You can go (buy) long and (sell) short.
- Full proceeds of short sale.
- No bid-ask spread on the primary securities.
- No transaction costs (frictions).
- No market impact.

Risk Neutral Valuation:

Notice (an algebraic coincidence):

$$S\Delta + B = S\frac{c_u - c_d}{(u-d)S} + \frac{uc_d - dc_u}{(u-d)R}$$

$$= \frac{\left[\frac{R-d}{u-d}\right]c_u + \left[\frac{u-R}{u-d}\right]c_d}{R}$$

Price of the option can be expressed as a linear combination - summing to one - of the payoff, discounted at the risk free rate.

For example, in our case:

$$\pi = \left(\frac{R-d}{u-d}\right) = \left(\frac{2-0.5}{3-0.5}\right) = 0.6$$

$$1-\pi = \left(\frac{u-R}{u-d}\right) = \left(\frac{3-2}{3-0.5}\right) = 0.4$$

and the option price is:

$$30 = \frac{0.6(100)}{2}$$

Actuarial vs. Financial (RN) Probability Measure

Question: What value of *p*, would make you indifferent between the stock and the bond if you were risk-neutral in utility:

Answer: $\pi Su + (1-\pi)Sd = SR$

implies: $\pi = \frac{R-d}{u-d}$

Practical Statement

If I were risk averse - which I am - I would only invest in the risky asset, if the actuarial probability p, of an up-move is greater than financial, also known as Risk Neutral probability, π.

Risk Neutral Valuation:

No Arbitrage price of contingent-claim is:

$$X_0 = \frac{(\pi) X_u + (1-\pi) X_d}{R}$$

which is the risk neutral expectation, discounted by the risk-free rate.

REMEMBER:

- RNV is a shortcut to avoid manufacturing the contingent claim.
- It only makes sense if the market is complete and the contingent claim can be manufactured.
- You price risk-neutrally because you should not be compensated for taking this financial risk. It is diversifiable, or non-systematic risk.

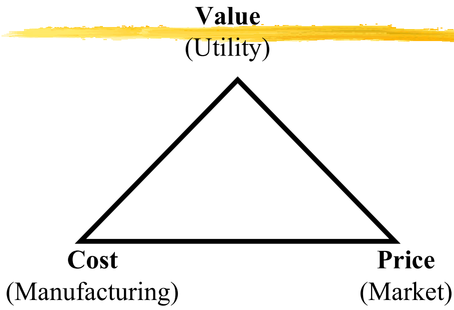
1. Risk Neutral: $S = \frac{\pi Su + (1-\pi)Sd}{R}$

2. Risk Premium: $S = \frac{pSu + (1-p)Sd}{R + \lambda\sigma}$

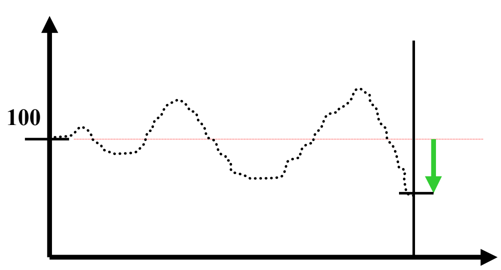
3. Certainty Equivalent: $S = \frac{pSu + (1-p)Sd - Z}{R}$

For example, if $p = 0.8 > \pi = 0.60$, then, $\lambda\sigma = 0.5$, or $Z = 50$.

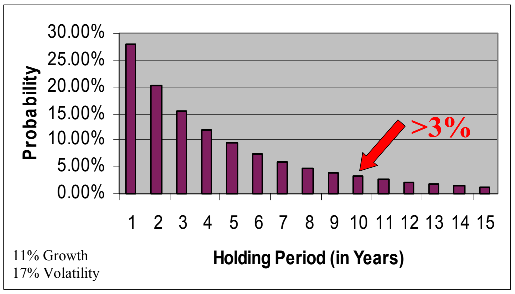
What is the Liability Worth?



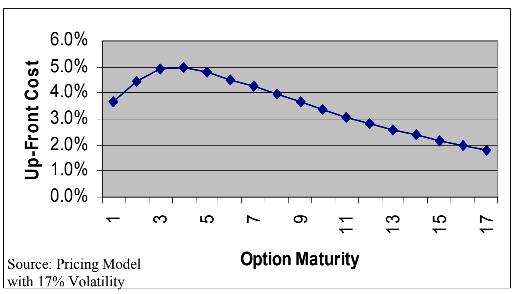
What is a Put Option Worth?



What is the probability of a negative holding-period return?



The cost of put-protection declines with time



Insurance Actuary vs. Financial Engineer: The 'worth' of a European-style put option

$r = 0.06$, $\mu = 0.1098$, $\sigma = 0.1871$

Valuation Method	T = 5 years	T = 10 years
RNV	4.93	3.46
QRM(90)	5.39	0.00
QRM(95)	14.75	2.33
CTE(90)	16.81	6.62
CTE(95)	23.79	13.05

QRM(x) is the sum needed, in a risk-free account, to pay the benefit x% of the time. CTE(x) is the sum needed to pay for the worst (1-x)% of the cases.

Case Study:

- Variable Annuity (VA) contracts in the US are like mutual funds with complete asset allocation control, with a guaranteed minimum death benefit (GMDB).
- Various types of GMDB, which range from return of premium (only) to a guaranteed return of x% per annum, up to ratchet.
- Question: How should we price, reserve, and set capital for these hybrid insurance and financial contracts?

Example of VA Cash Flow…

Year	Starting	Savings	Ending
1	0	50	55
2	55	50	112
3	112	50	178
4	178	50	215
5	215	50	230
6	230	50	335

GMDB has Value

What is the size of market?

- Between $800 - $1,000 billion in VAs.
- Best selling products have optional GMDB, resets and enhanced earnings GMDB.
- No two companies have the same combination of resets, roll-ups, ratchets, optional rides and age restrictions.
- Very heterogeneous commodity.

How are North American Companies Managing This Risk?
Anecdotal Evidence

- In total, 6-8 are hedging with derivatives
- Additional 10-12 are 'strongly considering'.
- Leading re-insurer has 15+ clients.
- Vast majority running 'naked'.
- Two examples:
  - Skandia: Using OTM-put to hedge -20% loss
  - SunLife: Dynamic hedging.

Moody's Investors Services: October 2000

- Reserving does not eliminate all risk.
- Capital requirement calculation is inadequate.
- Insurers will need to post extra statutory reserves and capital to offset exposures as markets fall, which will cause *surplus strain.*
- Companies should price and allocate capital to stochastic results at the 95% to 98% level.
- Lapses have not generally related to the efficient exercise of contract holder options, but this might change over time if advanced option pricing technology becomes available to contract holders.
- Interested in seeing results of companies stochastic product testing.

Canadian Institute of Actuaries

- Taskforce to raise awareness of risks.
- Examine Three Aspects:
  - Policy Liabilities.
  - Minimum Capital (Canadian MCCSR)
  - Capital Resiliency Testing (DCAT)
- Total Balance Sheet Requirements:
  - TBSR = Fund Value x Unique Product Factors
  - Capital = Max [0, TBSR - Policy Liabilities]
  - Factor credits for re-insurance and hedging

Other CIA Recommendations…

- Narrow the range of actuarial practice
- Stochastic Modeling for Policy Liabilities
- Factor approach for minimum capital
- Sample-paths should not be mandated
- At least 1,000 simulations.
- Use P-measure, not Q-measure
- Calibrate tails, not the dataset.
- Introduction of CTE(x%) measure.

American Academy of Actuaries:

C3 Phase 2 Project

- Recommendation for setting regulatory RBC (risk-based capital) requirements for VA guarantees.
- Stochastic scenarios using a calibrated fund performance distribution function for entire book on aggregate basis.
- For each yearly scenario, statutory surplus is calculated, reflecting statutory reserves, taxes and expenses.
- For each yearly scenario, the point in time with the PV of greatest statutory loss is chosen, and all PVs are tabulated.
- They recommend using CTE(90).
- Reinsurance and hedging will be reflected in the modeling

Scholarly Research

- Brennan, Schwartz and Boyle (1978)
- Windcliff, Forsyth, Vetzal and Morland
  - Series of papers at the University of Waterloo
  - Focus on the PDEs from R.N. valuation
- Boyle, Wirch, Hardy
  - Focus on reserving for Seg Funds
- Milevsky, Posner and Salisbury
  - Empirical and Theory on VA-GMDBs

Risk-Neutral Valuation:
Assumptions

- Examine on a per-policy at-issue basis.
- European-style static valuation.
- No lapses allowed. (Irrational?)
- Surrender Charge can not be used.
- Capital markets are complete.
- The yield curve is flat and non-dynamic.
- Mortality charge that funds the GMDB.

Risk Neutral Cost of a GMDB:
Return-of-Premium Guarantee

Age:	Female	Male
30	0.3 b.p.	0.4 b.p.
40	0.8 b.p.	1.3 b.p.
50	2.0 b.p.	3.5 b.p.
60	5.0 b.p.	8.7 b.p.
65	7.6 b.p.	13.0 b.p.

Assuming: 6% Interest, 20% Market Volatility, and 1994 GAM, Termination at age 75.

GMDB Vega: *Sensitivity to Volatility*
Return-of-Premium Guarantee: Age 50.

Age:	Female	Male
σ = 10%	0.5 b.p.	0.03 b.p.
σ = 15%	0.7 b.p.	1.2 b.p.
σ = 20%	2.0 b.p.	3.5 b.p.
σ = 30%	6.0 b.p.	10.4 b.p.
σ = 50%	14.0 b.p.	25.6 b.p.

Assuming: 6% Interest, 20% Market Volatility, and 1994 GAM, Termination at age 75.

Risk Neutral Cost of a GMDB:
5% Rising Floor Guarantee

Age:	Female	Male
30	1.7 b.p.	3.2 b.p.
40	4.4 b.p.	7.9 b.p.
50	10.8 b.p.	19.2 b.p.
60	21.6 b.p.	37.5 b.p.
65	22.5 b.p.	39.3 b.p.

Assuming: 6% Interest, 20% Market Volatility,
and 1994 GAM, Termination at 75, 200% cap on rising floor.

Risk Neutral Cost of a GMDB:
Look-back (Ratchet) Guarantee

Age:	Female	Male
30	15.1 b.p.	25.0 b.p.
40	18.9 b.p.	31.6 b.p.
50	24.6 b.p.	41.8 b.p.
60	32.8 b.p.	56.4 b.p.
65	36.1 b.p.	62.5 b.p.

Assuming: 6% Interest, 20% Market Volatility,
and 1994 GAM Termination at 75, continuous-time model.

How much more expensive are the enhanced GMDBs compared to the generic ones?

Age:	Female	Male
30	50 x	63 x
40	23 x	24 x
50	12 x	12 x
60	7 x	6 x
65	5 x	5 x

The younger the person, the more expensive the enhancement on a relative basis. At older ages the marginal cost is lower.

Main Message

- The GMDBs on Variable Annuities are not homogenous commodities.
- Many (older) VA-GMDBs are worthless, but a growing (recent) number of products are potential liabilities, if we value using (RN) capital market techniques.
- The contrast between insurance-actuarial and financial-engineering pricing.

Moving from Individuals to Books

- In practice options can not be priced policy-by-policy for individuals.
- The RNV method may not be relevant if we are not hedging.
- The RNV method does not provide guidance on the long-term Value-at-Risk.
- What is the distribution of cash-flows?
- Monte Carlo Simulation is essential.

The Brownian Motion

1 Continuous process that starts at zero.
2 Stationary and independent increments.
3 Mean of zero, variance of time.
4 Normal distribution at any point in time.

$$B_t \equiv N(0, \sqrt{t})$$

$$dB_t \approx \pm\sqrt{dt}$$

$$\Delta B_t \approx N(0, \sqrt{\Delta t})$$

Examples of diffusion processes that are driven by Brownian Motions

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

The Geometric Brownian Motion

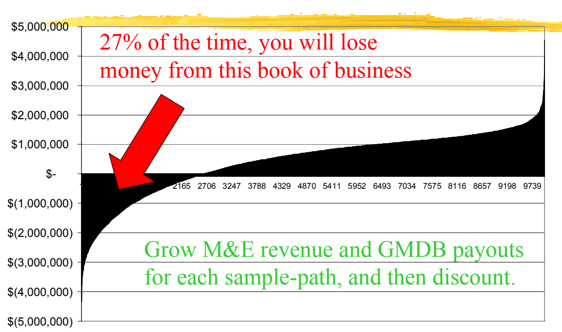
$$dr_t = \kappa(m - r_t)dt + \sigma\sqrt{r_t}\,dB_t$$

The Mean-Reverting Square-Root Process
C.I.R. 1985 (*Econometrica*)

Simulating C.I.R. 1985 Diffusion:
Monthly Approximation

r_i	ΔB_{i+1}	Δr_{i+1}
r_0	$N(0,\sqrt{1/12})$	$k(m-r_0)/12+\sigma\sqrt{r_0}\Delta B_1$
$r_1 = r_0 + \Delta r_1$	$N(0,\sqrt{1/12})$	$k(m-r_1)/12+\sigma\sqrt{r_1}\Delta B_2$
$r_2 = r_1 + \Delta r_2$	$N(0,\sqrt{1/12})$	$k(m-r_2)/12+\sigma\sqrt{r_2}\Delta B_3$
$r_3 = r_2 + \Delta r_3$	$N(0,\sqrt{1/12})$	$k(m-r_3)/12+\sigma\sqrt{r_3}\Delta B_4$

Present Value of P&L



How to Think About Tail Hedging

Assume: $100 Million VA-GMDB Portfolio

PV(P&L)

Mortality Charge	Prob. Loss	Expct. Loss
10 bp	22%	$5.7 M
20 bp	16%	$4.3 M
30 bp	11%	$3.6 M
40 bp	7%	$2.9 M
50 bp	3%	$1.4 M
65 bp (Fair Value)	0% (Perfect Hedge)	N.A.

Assumptions: 150 Total M&E; 25% Vol.; 100% SP.Cor.; 10% Lps; 1994 GAM, 5.5% Risk.Free; 12% Growth; Age 80 Termination.

What have we done?

- There are (at least) three ways to define what something is worth.
- When the payoff can be replicated, the price is invariant to preferences and market forces.
- Many (new) products mix financial and actuarial risk.
- Their pricing, hedging and reserving depends on market completeness.

Next Week...

- How RISKY is the stock market?
- What is a proper portfolio for Me Inc.?
- Using shortfall probabilities to manage asset allocation for individual portfolios.
- How To Avoid Outliving Your Money.

How to Reach Me:

- Milevsky@YorkU.ca
- Tel: (416) 736-2100 ext: 66014
- Fax: (416) 736-5487

www.milevsky.com

Moshe A. Milevsky, Ph.D.

Moshe (pronounced: Mow-Sheh) is an Associate Professor of Finance at the Schulich School of Business at York University, and the Executive Director of the Individual Finance and Insurance Decision (IFID) Centre, in Toronto, Canada. He has a Ph.D. in Finance (1996), an M.A. in Mathematics and Statistics (1992) and a B.A. in Mathematics and Physics (1990).

The focus of his teaching, research and consulting work is in the interplay between risk management, personal finance and insurance. In addition to many keynote lectures, and teaching his Ph.D., MBA and BBA courses at York University, he has worked as a risk management consultant and in Research & Development programs for a variety of North American companies in the financial services sector.

His consulting clients have include ScotiaMcleod, Trimark, Investors Group, Merrill Lynch, Bank of Montreal, CIBC-Wood Gundy, Manulife Financial, American International Group, AXA Group, Keyport Insurance, Ibbotson Associates, TIAA-CREF, National Life, amongst many other companies. In addition, he has provided expert testimony in a variety of court proceedings in the U.S. and Canada.

Prof. Milevsky has received numerous awards and research grants. They include grants from the Canadian Social Science and Humanities Research Council, The Society of Actuaries, The International Certified Financial Planner Board of Standards and The Canadian Investment Review Academic Sponsorship Program. In 1996 he was awarded The Financial Research Foundation's best Ph.D. Dissertation in Finance award and the American Association of Individual Investors award for best paper in Investment Theory.

Prof. Milevsky has published over 25 scholarly research articles in the *North American Actuarial Journal*, *The Journal of Risk and Insurance*, *Insurance: Mathematics and Economics*, *RISK Magazine*, *The Journal of Financial and Quantitative Analysis* and the *Journal of Derivatives*. He is the co-editor of the *Journal of Pension Economics and Finance* (Cambridge University Press).

He is the author of the Canadian best seller *Money Logic: Financial Strategies for the Smart Investor* (Stoddart, 1999), and the U.S. book *The Probability of Fortune: Financial Strategies with the Best Odds* (Stoddart, 2000). His innovative research has been cited in *Business Week, The Wall Street Journal, The New York Times*, *Barron's*, *Fortune* and *Money Magazine,* and he currently writes a monthly column for the *National Post* Business magazine.

Prof. Milevsky is an avid soccer player and opera connoisseur, and currently lives in Toronto with his wife Edna and three daughters, Dahlia, Natalie and Maya. You can visit Prof. Milevsky’s webpage at: www.milevsky.com

CAMBRIDGE
UNIVERSITY PRESS

Beginning Publication January 2002

Journal of Pension Economics and Finance

The Journal of Pension Economics and Finance is the first academic journal focusing on the economics and finance of pensions, retirement income and ageing. The Journal will be published three times per year and online. The journal is associated with the new OECD International Network of Pension Regulators and Supervisors which has over 100 member organizations in over 60 countries. Initial subscription prices are £20 per year for individuals and £80 per year for institutions.

The journal will publish original academic research papers on topics such
such as:

- Pensions and labour markets
- Macroeconomics of pensions
- Pension fund management, including asset-liability modeling
- Regulation of pension plans
- Valuation of corporate and social security pension liabilities.
- Corporate governance issues relevant to pensions
- Political economy and public choice models of pensions and social security
- Financial planning and the decisions of individuals about pension arrangements
- Marketing of private pensions

To subscribe, please fill out the form on the attached page or visit us on the web at http://www.pensions-journal.com.