# FINE6860: Lecture #7
# Models for D.B. vs. D.C.

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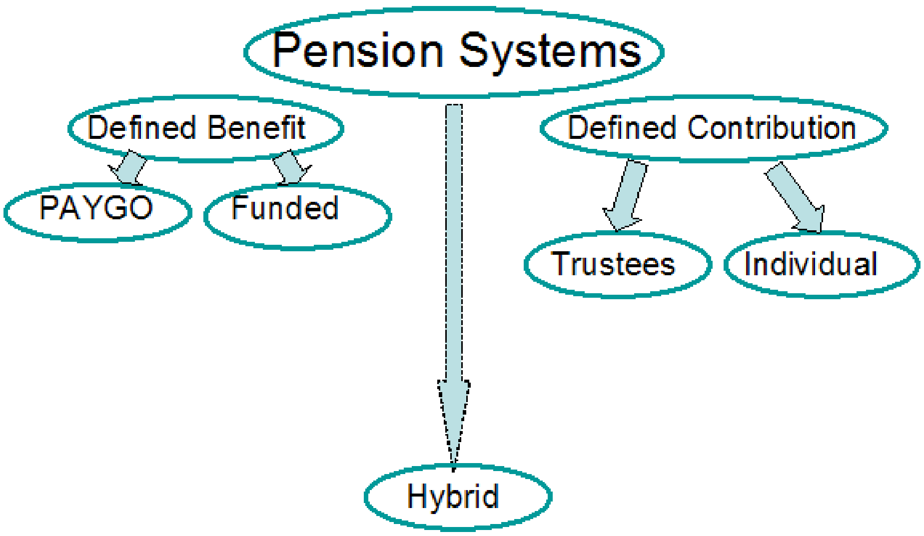



# 1 Understanding Your Choice of Pensions

# 2 The Core of the Defined Contribution Pension.

I will start with at the very end by displaying the two main equations for D.C. and D.B. pension plans.

When you retire from a D.C. pension – at age $x$ or after $T$ years of employment and participation in the plan – the annual retirement income (a.k.a. pension) you are entitled to is specified by the following formula:

$$\text{D.C. Pension Income} := \frac{\int_0^T c(s)e^{g(s)(T-s)}w(s)ds}{\overline{a}_x}, \qquad (1)$$

where $c(s)$ is the contribution rate, $g(s)$ is the "realized" investment growth rate and $w(s)$ is the wage or salary; all of which are parameterized by time $s$. Finally, $\overline{a}_x$ is the familiar pension annuity factor which converts a lump sum in the numerator into a periodic income flow. It is important to note that equation (1) is backward-looking and meant to be used at retirement to compute a retirement income benefit under the pension plan.

Allow me to do a basic example so that you can develop some intuition. Assume that you are just about to turn $x = 65$ years old, the point at which you will be retiring from the labour force and will start to draw a pension. You have been working for the same company for the last $T = 30$ years and have been earning a constant $w(s) = 50,000$ dollars during each and every year. Now, assume that each year your employer contributed $7\%$ of your salary to a Defined Contribution pension fund and that this fund earned $g(s) = 10\%$ during each of the 30 years. I now that most of this is highly unrealistic, but bear with me for a moment. In this case, the funds in your DC account will have accumulated to:

$$0.07 \int_0^{30} 50000 e^{(0.10)(30-s)} ds = \$667,994 \qquad (2)$$

dollars at retirement. Equation (2) "adds up" the 7% pension contribution plus investment gains for the entire 30-year period. Finally, the $\$667,994$ is divided into the $\overline{a}_{65} = 11.395$ pension annuity factor to yield a retirement income of $\$58,622$ per year. The pension annuity factor was obtained by using our favorite $m = 86.34$, $\beta_{21} = 9.5$ and $r = 5\%$ parameters. Note that this is a very "nice" pension. Your salary was $\$50,000$ per year, which means that your pension has replaced $58622/50000 = 117\%$ of your pre-retirement income.

Stepping back to time zero – as opposed to the age and time of retirement – your pension income is a random variable which can be expressed as:

$$\frac{\int_0^T c(s)e^{\mathbf{B}_{(T-s)}^{(\nu,\sigma)}}\mathbf{w}(s)ds}{\overline{a}_x}, \tag{3}$$

where $\mathbf{B}_t^{(\nu,\sigma)}$ is the familiar Brownian motion term I introduced in an earlier chapter, and $\mathbf{w}(s)$ denotes the random and unpredictable wage or salary over the $T$ years of work. In fact, some might argue that even the denominator in equation (3) should be "bolded" to denote the fact that interest rates and perhaps even GoMa parameters are unknown so far in advance of retirement. This is a very legitimate point and we will return to the "stochasticity" of annuity factors later in the book.

# 3 The Core of a Defined Benefit Pension

In contrast to the D.C. pension formula, the D.B. formula focuses and provides a guarantee on actual retirement income. In the D.B. case there is no numerator or denominator, but rather a direct formula:

$$\text{D.B. Pension Income} := \alpha T \beta \int_0^T e^{-\beta(T-s)} w(s) ds, \tag{4}$$

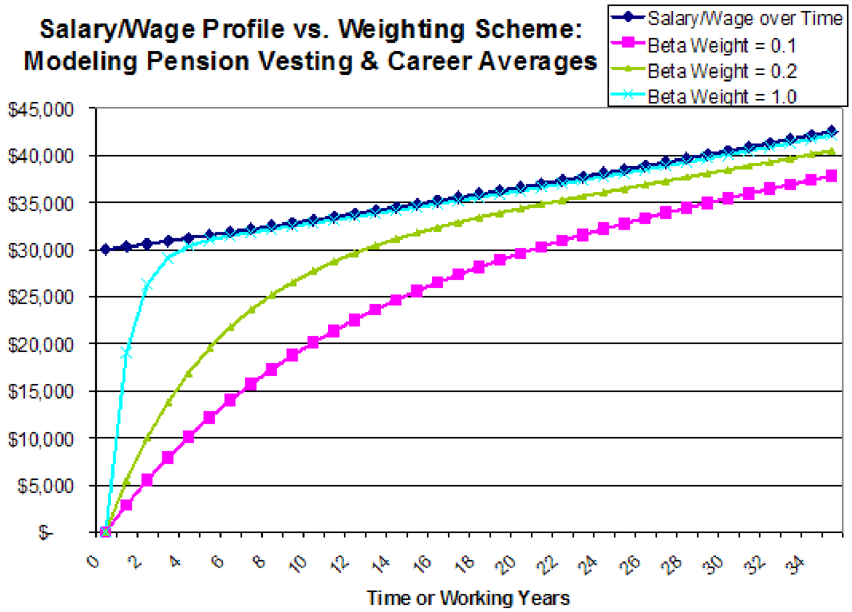
which I will abbreviate as:

$$\text{D.B. Pension Income} := \alpha T \omega(T), \tag{5}$$

where the new "salary weighting function" is defined by:

$$\omega(T) = \beta \int_0^T e^{-\beta(T-s)} w(s) ds. \tag{6}$$

Once again we have a number of moving parts so I will explain each term individually. Figure #2 provides a graphical illustration of the salary weighting function. The greater the value of $\beta$ the more weight is placed on "recent" or "final" wages versus the path of wages. For example, at a $\beta = 0.1$, the value of the $\omega(t)$ function starts quite low and the slowly increases over time as the wage increases. However, when $\beta = 1$ (and in theory it can go as high as infinity) the $\omega(t)$ function moves very quickly to a number close to $w(t)$. In some sense $\omega(t)$ and $w(t)$ are the same, after a while.




In fact, when the salary for wages satisfies as simple exponential growth equation:

$$w(t) = we^{kt}, \tag{7}$$

where $k$ is an annual growth rate, then the salary weighting function displayed and defined in equation (6) can be integrated explicitly to yield:

$$\omega(T) = \frac{\beta w}{\beta + k}\left(e^{kT} - e^{-\beta T}\right). \tag{8}$$

And, when $k = 0$ which is a flat wage profile, the function collapses to $w(1 - e^{-\beta T})$, which rapidly converges to the salary value itself as $e^{-\beta T}$ gets very small.

So, for example, if $w = 30000$ and it grows each year by $k = 1\%$, then when $\beta = 0.1$ equation (8) leads to a value of $\omega(30) = \$35,456$ and when $\beta = 0.2$ then $\omega(30) = \$38,497$

and finally when $\beta = 1$, then $\omega(30) = \$40,095$, which is extremely close to $30000e^{(0.01)(30)} = \$40496$, which is the actual salary at retirement. And, if the D.B. pension stipulates an accrual rate of $\alpha = 1\%$ for each year of employment, then at retirement the retiree will be entitled to a nominal pension income of $(30)(0.01)(35456) = \$10,637$ under a $\beta = 0.1$ weighting, $(30)(0.01)(38497) = \$11,549$ under a $\beta = 0.2$ weighting and $(30)(0.01)(40095) = \$12,028$ under a $\beta = 1.0$ weighting.

I have now reached the point where meaningful comparisons can be made between D.B. and D.C. plan benefits.

| Defined Contribution (D.C.) Retirement Income | | | |
|---|---|---|---|
| Initial Salary \$30,000 + 30 Years Work | | | |
| 1% Salary Growth -> \$40,496 at Retirement | | | |
| D.C. Rate | Assumed Investment Returns | | |
| Contribution | $g = 3\%$ | $g = 5\%$ | $g = 7\%$ |
| $c = 4\%$ | $\$5,105$ | $\$7,203$ | $\$10,452$ |
| $c = 6\%$ | $\$7,658$ | $\$10,805$ | $\$15,678$ |
| $c = 8\%$ | $\$10,210$ | $\$14,407$ | $\$20,904$ |
| $c = 10\%$ | $\$12,763$ | $\$18,009$ | $\$26,130$ |
| $c = 12\%$ | $\$15,315$ | $\$21,610$ | $\$31,356$ |
| Table #1: Source The IFID Centre | | | |
| $m = 86.34, b = 9.5, \lambda = 0, r = 3.5\%, k = 1\%$ | | | |

| D.C. Pension: Income Replacement Rate | | | |
|---|---|---|---|
| Initial Salary \$30,000 + 30 Years Work | | | |
| 1% Salary Growth -> \$40,496 at Retirement | | | |
| D.C. Rate | Assumed Investment Returns | | |
| Contribution | $g = 3\%$ | $g = 5\%$ | $g = 7\%$ |
| $c = 4\%$ | 12.6% | 17.8% | 25.8% |
| $c = 6\%$ | 18.9% | 26.7% | 38.7% |
| $c = 8\%$ | 25.2% | 35.6% | 51.6% |
| $c = 10\%$ | 31.5% | 44.5% | 64.5% |
| $c = 12\%$ | 37.8% | 53.4% | 77.4% |
| Table #2: Source The IFID Centre | | | |
| $m = 86.34, b = 9.5, \lambda = 0, r = 3.5\%, k = 1\%$ | | | |

| Defined Benefit (D.B.) Retirement Income | | | |
|---|---|---|---|
| Initial Salary \$30,000 + 30 Years Work | | | |
| 1% Salary Growth -> \$40,496 at Retirement | | | |
| D.B. Rate | Salary Weighting Scheme | | |
| Accrual | $\beta = 0.1$ | $\beta = 0.2$ | $\beta = 1$ |
| Rate | Avg = \$35,457 | Avg = \$38,497 | Avg = \$40,095 |
| $\alpha = 1.00\%$ | $\$10,637$ | $\$11,549$ | $\$12,028$ |
| $\alpha = 1.25\%$ | $\$13,296$ | $\$14,436$ | $\$15,036$ |
| $\alpha = 1.50\%$ | $\$15,955$ | $\$17,323$ | $\$18,043$ |
| $\alpha = 1.75\%$ | $\$18,615$ | $\$20,211$ | $\$21,050$ |
| $\alpha = 2.50\%$ | $\$26,592$ | $\$28,872$ | $\$30,071$ |
| Table #3: Source The IFID Centre | | | |
| Note: All Numbers are in nominal terms. | | | |

| D.B. Pension: Income Replacement Rate | | | |
|---|---|---|---|
| Initial Salary $30,000 + 30 Years Work | | | |
| 1% Salary Growth -> $40,496 at Retirement | | | |
| D.B. Rate | Salary Weighting Scheme | | |
| Accrual | $\beta = 0.1$ | $\beta = 0.2$ | $\beta = 1$ |
| Rate | Avg = $35,457 | Avg = $38,497 | Avg = $40,095 |
| $\alpha = 1.00\%$ | $26.3\%$ | $28.5\%$ | $29.7\%$ |
| $\alpha = 1.25\%$ | $32.8\%$ | $35.6\%$ | $37.1\%$ |
| $\alpha = 1.50\%$ | $39.4\%$ | $42.7\%$ | $44.5\%$ |
| $\alpha = 1.75\%$ | $46.0\%$ | $49.9\%$ | $52.0\%$ |
| $\alpha = 2.50\%$ | $65.6\%$ | $71.3\%$ | $74.3\%$ |
| Table #4: Source The IFID Centre | | | |
| Note that all ratios are in nominal terms. | | | |

# 4 What is the Current Value of the D.B. Pension?

Most of the previous discussion centered on retirement income and what you are entitled to once you are retired. I would like to step back from retirement by a few years. Imagine that you are $y$ years-old and have worked for $\tau$ years at your current job that offers a Defined Benefit pension plan, where $0 < \tau \leq T$ . The D.B. plan allows you to retire at age $x$, for example 65 years of age, so that $y - x = T - \tau$, by definition.

There are three possible ways to measure the current "value or worth" of what you are entitled to at retirement age $x$. The first measure of the firm's pension obligation to their employees, is called the Retirement Benefit Obligation (RBO), the second is the Projected Benefit Obligation (PBO) and the third is called the Accumulated Benefit Obligation (ABO). Here is the formal definition of all three quantities:

$$\Upsilon_y^{\text{RBO}} = e^{-r(x-y)} \alpha T \omega(T) \overline{a}_x, \quad (9)$$

$$\Upsilon_y^{\text{PBO}} = e^{-r(x-y)} \alpha \tau \omega(T) \overline{a}_x, \quad (10)$$

$$\Upsilon_y^{\text{ABO}} = e^{-r(x-y)} \alpha \tau \omega(\tau) \overline{a}_x. \quad (11)$$

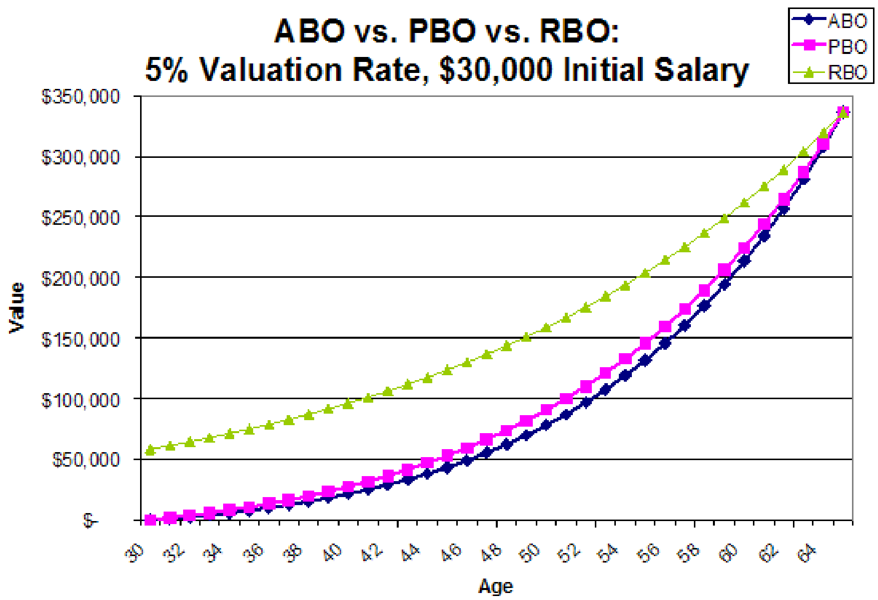



Figure #3 provides a graphical illustration of the relationship between the three possible measures of the value of the pension promise at age $y$. The underlying parameters for this particular figure are the standard $m = 86.34$ and $b = 9.5$ which leads to the pension annuity factor of $\overline{a}_{65} = 11.3949$ at retirement. The initial salary of $w = \$30,000$ grows by $k = 1\%$ each year until it reaches $w(35) = \$42,572$ at age $x = 65$. The salary weighting function under a $\beta = 1$ leads to $\omega(35) = \$42,151$. Finally, $\alpha = 2\%$ per each year of credited service in the D.B. plan. This leads to $(0.02)(35)(42151)(11.3949) = \$336,214$ at the lump-sum value at retirement. Using our notation, $\Upsilon_{65} = \$336,214$ for the RBO, PBO and ABO. This is the point (age) at which the three curves meet.

The distinction between ABO, PBO and RBO is very important and critical to understanding some of the accounting issues that arise. To get a better sense of the interaction between the values here are some numerical examples.

| You are 45 years-old with 15 years of pension service. | | | | |
|---|---|---|---|---|
| You earn $34,855 per year and plan to retire at age 65. | | | | |
| What is the Current Value of Your Retirement Pension? | | | | |
| Valuation Rate | ABO | PBO | RBO | $\overline{a}_{65}$ |
| $r = 5\%$ | $43,399 | $53,008 | $123,685 | 11.394 |
| $r = 7\%$ | $24,686 | $30,152 | $70,355 | 9.669 |
| $r = 9\%$ | $14,271 | $17,431 | $40,672 | 8.339 |
| Table #5: Source IFID Centre Calculations | | | | |
| $m = 86.34, b = 9.5, k = 1\%, \alpha = 2\%, \beta = 1$ | | | | |

# 5 Pension Funding and Accounting

| One Year Later...Now 46 Years Old | | | |
|---|---|---|---|
| How Much Has the Pension Value Changed? | | | |
| Valuation Rate | $\Delta$ABO | $\Delta$PBO | $\Delta$RBO |
| $r = 5\%$ | \$5,756 | \$6,433 | \$6,341 |
| $r = 7\%$ | \$3,839 | \$4,342 | \$5,101 |
| $r = 9\%$ | \$2,552 | \$2,913 | \$3,830 |
| Table #6: Source IFID Centre | | | |
| $m = 86.34, b = 9.5, k = 1\%, \alpha = 2\%, \beta = 1$ | | | |

| Age 30 Starting Salary of \$30,000 | | | |
|---|---|---|---|
| Fixed Valuation Rate of 5% | | | |
| How Much Has the Pension Value Changed? | | | |
| Age Ending | ΔABO | ΔPBO | ΔRBO |
| $y = 35$ | \$2,012 | \$2,562 | \$3,659 |
| $y = 45$ | \$5,252 | \$5,947 | \$6,032 |
| $y = 55$ | \$12,640 | \$12,646 | \$9,945 |
| $y = 65$ | \$28,626 | \$25,535 | \$16,397 |
| Table #7: Source IFID Centre | | | |
| $m = 86.34, b = 9.5, k = 1\%, \alpha = 2\%, \beta = 1$ | | | |

| A Closer Look at the Projected Benefit Obligation (P.B.O.) | | | | | |
|---|---|---|---|---|---|
| How Does it Change from the Prior Year? | | | | | |
| Age | Salary | Int. Cost | + Svc. Cost | = $\Delta$PBO | Svc. % Sly. |
| $y = 35$ | \$31,538 | \$418 | \$2,143 | \$2,562 | 6.80% |
| $y = 45$ | \$34,855 | \$2,413 | \$3,534 | \$5,947 | 10.14% |
| $y = 55$ | \$38,521 | \$6,820 | \$5,826 | \$12,646 | 15.13% |
| $y = 65$ | \$42,572 | \$15,929 | \$9,606 | \$25,535 | 22.56% |
| Table #8: IFID Centre: $k = 1\%, \alpha = 2\%, \beta = 1, r = 5\%$ | | | | | |

| A Closer Look at the Accumulated Benefit Obligation (A.B.O.) | | | | | |
|---|---|---|---|---|---|
| How Does it Change from the Prior Year? | | | | | |
| Age | Salary | Int. Cost | + Svc. Cost | = $\Delta$ABO | Svc. % Sly. |
| $y = 35$ | \$31,538 | \$301 | \$1,711 | \$2,012 | 5.42% |
| $y = 45$ | \$34,855 | \$1,956 | \$3,296 | \$5,252 | 9.46% |
| $y = 55$ | \$38,521 | \$6,109 | \$6,531 | \$12,640 | 16.95% |
| $y = 65$ | \$42,572 | \$15,770 | \$12,856 | \$28,626 | 30.20% |
| Table #9: IFID Centre: $k = 1\%, \alpha = 2\%, \beta = 1, r = 5\%$ | | | | | |