FINE6860: Lecture #2
Models of Mortality

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1 Mortality Tables and Rates

A mortality table – perhaps better referred to as vector – maps an age group (x) into a probability of death q_x, within the next year. By definition, $0 \leq q_x \leq 1$ and $q_N = 1$, for some large enough $N \approx 110$.

2 Conditional Probability of Survival

If an individual is currently aged x, then the probability of surviving to age n is denoted and defined by:

$$({}_n p_x) = \prod_{i=0}^{n-1} (1 - q_{x+i}) \tag{1}$$

Discuss: Why is this the correct formula?

3 Remaining Lifetime Random Variable

Introduce a random variable (r.v.) denoted by $\mathbf{T}_x$ and indexed by age x, which represents the *remaining lifetime* for an individual currently aged x. By definition, the cumulative distribution function (CDF) is:

$$({}_tp_x) := 1 - F_x(t) = \Pr[\mathbf{T}_x > t] \tag{2}$$

The function $({}_tp_x)$ is the probability of surviving to t. The function $F_x(t)$ is the probability of dying before t. They both must add-up to one.

The remaining lifetime random variable $\mathbf{T}_x$ has a probability density function (PDF) denoted and defined by:

$$F_x(t) = \int_0^t f_x(s)ds, \tag{3}$$

when the random variable $f_x(t)$ is continuous, or

$$F_x(n) = \sum_{i=1}^{n} \Pr[\mathbf{T}_x = x_i], \tag{4}$$

when the random variable $\mathbf{T}_x$ is discrete.

Here x_i denotes the ages at which people are "allowed" to die. For example, $x_1 = 70$, $x_2 = 85$ and $x_3 = 95$. In this case $\mathbf{T}_x = \{70, 85, 90\}$ and the probability mass function (PMF) is instead of the PDF. A possible example would be:

$\mathbf{T}_{60}$	$\Pr[\mathbf{T}_{60} = x_i]$
70	8/12
85	3/12
95	1/12

The expected value of the remaining lifetime r.v. would be:

$$E[\mathbf{T}_{60}] = \frac{8}{12} \times 10 + \frac{3}{12} \times 25 + \frac{1}{12} \times 35 = 15.833 \text{ years}$$

Note also that:

$$\begin{aligned}
\Pr[\mathbf{T}_{60} &\leq 10] = 8/12 \\
\Pr[\mathbf{T}_{60} &\leq 25] = 8/12 + 3/12 = 11/12 \\
\Pr[\mathbf{T}_{60} &\leq 25] = 8/12 + 3/12 + 1/12 = 1
\end{aligned}$$

4 Instantaneous Force of Mortality

As long as $({}_tp_x)$ is constant or decreasing with respect to t, then I am entitled to represent this function using the following:

$$({}_tp_x) = e^{-\int_x^{x+t} \lambda(s)ds}, \tag{5}$$

where the curve $\lambda(s) \geq 0$, for all $s \geq 0$. Think of $\lambda(s)$ as the instantaneous rate of death at age s. When $s = 0$, then $({}_sp_x) \to 1$, and when $s \to \infty$, we must have that $({}_sp_x) \to 0$ so that $\int_x^{x+\infty} \lambda(s)ds \to \infty$.

Note that by a simple change-of-variables $u = s - x$, we can re-write equation equation (5) as:

$$({}_tp_x) = e^{-\int_0^t \lambda(x+u)du}. \tag{6}$$

Integrating the curve $\lambda(s)$ from a lower bound of x to an upper bound $x + t$ is mathematically equivalent to integrating the curve starting at $\lambda(x + s)$ from a lower bound of 0 to an upper bound of t.

Note some facts about "mortality" now that we have defined $({}_tp_x)$ in this (more restrictive) way. Take derivatives of both sides:

$$\frac{\partial}{\partial t}({}_tp_x) = -({}_tp_x)\lambda(x+t) \tag{7}$$

Therefore

$$f_x(t) = (1 - F_x(t))\lambda(x+t) \tag{8}$$

5 Using the ODE Relationship:

Based on the Ordinary Differential Equation (ODE) for the function $F_x(t)$ displayed above, we can represent the IFM as:

$$\lambda(x+t) = \frac{f_x(t)}{1 - F_x(t)}, \qquad t \geq 0. \tag{9}$$

Note that $F_x(t) \to 1$ as $t \to \infty$ (everyone dies eventually) and therefore $\lambda(t) \to \infty$ as $t \to \infty$, unless $f_x(t) \to 0$ in the numerator "faster".

Note also the relationship implied by equation (9) leads to:

$$F_x(t) = 1 - \frac{f_x(t)}{\lambda(x+t)}, \tag{10}$$

which then implies:

$$f_x(t) = ({}_tp_x)\lambda(x+t). \tag{11}$$

In sum, the above relationships allow us to "create" mortality laws in two different ways.

- We can start with a CDF $F_x(t) = 1 - ({}_tp_x)$ and then take the derivative to create the PDF $f_x(t)$ and then use equation (9) to get the IFM $\lambda(x)$.
- Or, alternatively, we can start with the IFM and build the CDF $F_x(t) = 1 - ({}_tp_x)$ using equation (5) and then take derivatives to arrive at the PDF $f_x(t)$.

Question: Using some of the qualitative features we would expect from the IFM curve, can we use $\underline{\text{any}}$ functional form for $f_x(t)$ and $F_x(t)$ or are there some natural restrictions on the remaining lifetime random variable?

Example: The normal distribution we are all familiar with. In this case the CDF is:

$$\mathbf{N}(m,b,t) = \int_{-\infty}^{t} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-m}{b}\right)^2} dz \tag{12}$$

Is negative feasible? Let us plot the ratio and see how it looks like. Excel spreadsheet.

Table #2 (Normal Odds), Figure #2a, Figure #2b (CDF, PDF, Hazard).

6 Moments in Your Life

We can now define the concept of moments and then move-on to life expectancy and standard deviation of remaining lifetime.

$$E[\mathbf{T}_x] = \int_0^\infty t f_x(t) dt \tag{13}$$

When $\mathbf{T}_x$ is a discrete random variable, the definition is:

$$E[\mathbf{T}_x] = \sum_{i=1}^{N} x_i \Pr[T_x = x_i] \tag{14}$$

Note that this is equivalent to:

$$E[\mathbf{T}_x] = \int_0^\infty ({}_t p_x) dt \tag{15}$$

Why? Can you prove it.

The second moment is:

$$E[\mathbf{T}_x^2] = \int_0^\infty t^2 f_x(t) dt \tag{16}$$

Remember the definition of standard deviation:

$$SD[\mathbf{T}_x] = \sqrt{E[\mathbf{T}_x^2] - E^2[\mathbf{T}_x]}. \tag{17}$$

The variance of a random variable, is the second moment minus the first moment squared.

7 Median Remaining Lifetime

Distinct from the above, the median remaining lifetime is:

$$\Pr[\mathbf{T}_x < M[\mathbf{T}_x]] = 0.5$$

The median remaining lifetime (MRL) will be less than the expected remaining lifetime (ERL) in all cases. Why?

8 Exponential Law of Mortality

Assume that the IFM curve satisfies $\lambda(x+t) = \lambda$, which is a constant across all ages and times. In this case, let us "build" the $F_x(t)$ and $f_x(t)$ functions using equation (5).

Note that we have:

$$({}_t p_x) = e^{-\int_x^{x+t} \lambda ds} = e^{-\lambda t}. \tag{18}$$

The integral in the exponent collapses (i.e. can be solved to produce) a linear function λt. In this case the current age x does not really impact the probability of survival since all that matters is the magnitude of the IFM, λ.

A number of mathematical objects "fall" into our lap:

$$\begin{aligned} F_x(t) &= 1 - e^{-\lambda t}, \\ f_x(t) &= \lambda e^{-\lambda t}. \end{aligned} \tag{19}$$

Table #3 can be contrasted with Table #2. It provides the Exponential Odds together with Figure #3a, Figure #3b.

The expected remaining lifetime (ERL) is:

$$E[\mathbf{T}_x] = \int_0^\infty t\lambda e^{-\lambda t} dt = \frac{1}{\lambda}. \tag{20}$$

For example, when $\lambda = 0.10$, the ERL is $E[T_x] = 10$ and when $\lambda = 0.05$ the ERL is $E[T_x] = 20$. In contrast, the median remaining lifetime (MRL) is obtained by solving:

$$\frac{1}{2} = e^{-\lambda M[\mathbf{T}_x]} \Longleftrightarrow M[\mathbf{T}_x] = \frac{\ln[2]}{\lambda} < \frac{1}{\lambda}. \tag{21}$$

For example, when $\lambda = 0.05$ the MRL is $M[T_x] = \ln[2]/0.05 = 13.862$ years, in contrast to the ERL of $1/0.05 = 20$ years. Notice the 6 years gap between the two.

9 Gompertz-Makeham Law of Mortality

Like in the case of the exponential law of mortality, the Gompertz-Makeham (GM) law of mortality is "built" using the IFM curve $\lambda(x)$. In the GM case, the definition is:

$$\lambda(x) = \lambda + \frac{1}{b} e^{(x-m)/b}, \quad t \geq 0 \tag{22}$$

The instantaneous force of mortality is a constant λ plus a time-dependent exponential curve. This curve increases with age and goes to infinity at $t \to \infty$.

When the individual is exactly $x = m$ years-old, the GM-IFM curve is $\lambda(m) = \lambda + 1/b$, but when the individual is younger ($x < m$) the GM-IFM curve is $\lambda(x) < \lambda + 1/b$ and when the individual is older ($x > m$) the GM-IFM curve is $\lambda(x) > \lambda + 1/b$. Thus, $x = m$ is a special age point on the IFM curve. More on this later.

The convention is to label equation (22) the Gompertz-Makeham law when $\lambda > 0$ and Gompertz alone, when $\lambda = 0$. In the Gompertz case, typical numbers for the parameters are $m = 82.3$ and $b = 11.4$, under which $\lambda(65) = 0.01923$ and $\lambda(95) = 0.26724$.

Note that sometime I use an alternative representation for the

GM law:

$$\lambda(t) = h + h_1 e^{h_2 t}, \quad t \geq 0. \tag{23}$$

A casual inspection will reveal the equivalence between the two functional forms of the GM law, since $h = \lambda$, $h_2 = 1/b$ and $h_1 = \exp\{-m/b\}/b$. Alternatively, we can move from the (λ, m, b) specification to the (h, h_1, h_2) specification by the transformation $b = 1/h_2$ and $m = -\ln[h_1/h_2]/h_1$. Both of the specifications will be used alternatively.

By the construction specified in equation (5), the conditional probability of survival under the GM-IFM curve is equal to:

$$\begin{aligned}
({}_tp_x) &= e^{-\int_x^{x+t}\left(\lambda+\frac{1}{b}e^{(s-m)/b}\right)ds} && (24)\\
&= e^{-\lambda t+b(\lambda(x)-\lambda)(1-e^{t/b})}, && (25)
\end{aligned}$$

and $F_x(t) = 1 - ({}_tp_x)$.

For example, when $\lambda = 0$, $m = 82.3$ and $b = 11.4$ equation (24) results in $F_{65}(20) = 0.6493$ and $F_{65}(10) = 0.2649$ as well as $F_{75}(30) = 0.9988$.

By taking derivatives of $F_x(t)$ with respect to t, we recover the probability density function (PDF) of the remaining lifetime random variable $f_x(t) = F'_x(t)$, which is left as an assignment problem.

We can take the "easy" route by appealing to equation (11) which leads us to:

$$\begin{aligned} f_x(t) \;=\; & e^{-\lambda t+b(\lambda(x)-\lambda)(1-e^{t/b})} \\ & \times \left(\lambda+\frac{1}{b}e^{(x-m)/b}\right), \end{aligned} \tag{26}$$

which is the $({}_tp_x)$ of the Gompertz-Makeham law multiplied by the IFM curve.

The Expected Remaining Lifetime (ERL) under the Gompertz-Makeham law of mortality is:

$$\begin{aligned} E[\mathbf{T}_x] \;=\; & \int_0^\infty e^{-\lambda t+b(\lambda(x)-\lambda)(1-e^{t/b})}dt \\ =\; & \frac{b\Gamma(-\lambda b, b(\lambda_x-\lambda))}{e^{(m-x)\lambda+b(\lambda-\lambda_x)}}, \end{aligned} \tag{27}$$

where the notation:

$$\Gamma(a,c)=\int_c^\infty e^{-t}t^{(a-1)}dt,$$

is the incomplete Gamma function.

The following tables provide numerical examples under a variety of values for m, b.

Female: 1994 GAM Table			
Age (x)	m	b	$x + E[\mathbf{T}_x]$
30	88.8379	9.213	83.61
40	88.8599	9.160	83.82
50	88.8725	9.136	84.21
60	88.8261	9.211	84.97
65	88.8403	9.183	85.69

And, for males:

Male: 1994 GAM Table			
Age (x)	m	b	$x + E[\mathbf{T}_x]$
30	84.4409	9.888	78.94
40	84.4729	9.831	79.31
50	84.4535	9.922	79.92
60	84.2693	10.179	81.17
65	84.1811	10.282	82.25

Note that I have used different m, b values at different ages. Why?

Under the alternative parametrization $\lambda(t) = h + h_1 e^{h_2 t}$, it is relatively easy to show that:

$$F(t) = 1 - \exp\{-\frac{h_1}{h_2} e^{h_2 t} - ht + \frac{h_1}{h_2}\} \tag{28}$$

and:

$$\begin{aligned} f(t) \ &= \ (h + h_1 e^{h_2 t}) \\ &\quad \times \exp\{-\frac{h_1}{h_2} e^{h_2 t} - ht + \frac{h_1}{h_2}\} \end{aligned} \tag{29}$$

10 General Hazard Rate

For example, it is common to model the rate at which people "lapse or surrender" an insurance, annuity or pension contract using the curve:

$$\gamma(t) = \gamma - \frac{\gamma_1}{t + \gamma_2}, \quad t > 0, \tag{30}$$

where $\gamma \geq 0, \gamma_1 \geq 0$ and $\gamma_2 > 0$. The hazard rate starts off at time zero at a value of $\gamma - \gamma_1/\gamma_2$ and then increases at a rate of $\gamma_1/(t + \gamma_2)^2$ until it asymptotes to a value γ. Thus, for this hazard rate function to be positive and well defined we must impose the additional restriction that $\gamma - \gamma_1/\gamma_2 > 0$.

From the construction provided by equation (9), we have that:

$$\gamma(t) = \frac{g(t)}{1 - G(t)}, \tag{31}$$

where $G(t) = \Pr[\mathbf{L} \leq t]$ is the cumulative distribution function (CDF) and $g(t) = G'(t)$ is the probability density function (PDF) of the random variable $\mathbf{L}$.This leads to the following solution

$$G(t) = 1 - e^{-\int_0^t \gamma(s)ds}. \tag{32}$$

Some algebra and calculus leads to:

$$\begin{aligned} G(t) &= 1 - \exp\{-\int_0^t \gamma ds\} \exp\{\int_0^t \frac{\gamma_1}{s+\gamma_2} ds\} \\ &= 1 - \exp\{-\gamma t\} \left(\frac{t}{\gamma_2} + 1\right)^{\gamma_1}. \end{aligned} \tag{33}$$

This expression obviously collapses to $1 - \exp\{-\gamma t\}$ when $\gamma_1 = 0$. Finally, the PDF for the future lapse-time random variable can be written explicitly as:

$$\begin{aligned} g(t) &= \left(\gamma - \frac{\gamma_1}{t+\gamma_2}\right) \\ &\quad \times \left(\exp\{-\gamma t\} \left(\frac{t}{\gamma_2} + 1\right)^{\gamma_1}\right). \end{aligned} \tag{34}$$

Once again we have used the convenient relationship between the CDF, PDF and IHR.

Question to ponder: People purchase life insurance and must pay premiums on a regular basis. Assume they "lapse" or "stop paying" their insurance premiums at a rate of:

$$\gamma(t) = 0.10 + \frac{0.09}{t+1}.$$

The instantaneous force of mortality (IFM) is Gompertz with parameters $m = 82.3$ and $b = 11.4$, so that:

$$\lambda(x) = \frac{1}{11.4} e^{\left(\frac{x-82.3}{11.4}\right)}.$$

What is the probability this person will die while the insurance policy is "unlapsed" or in coverage?

11 Working with Joint Lifetimes

The following table provides (reasonable) survival probabilities at age 65, using the Gompertz law of mortality under parameters $m = 88.18$ and $b = 10.5$ for males, and $m = 92.63$ and $b = 8.78$ for females.

Age 65	**Probability**	
Survive To:	Male	Female
70	0.935	0.967
75	0.839	0.912
80	0.705	0.823
85	0.533	0.686
90	0.339	0.497
95	0.164	0.281
100	0.023	0.103

These number come directly from equation (24) under $x = 65$. Imagine a married couple, both aged 65 who are interested in computing joint survival probabilities.

Question: What is the probability that they both survive from the current age 65 to age 90?

Answer:

$$\begin{aligned} & \left({}_{25}p_{65}^{\text{male}}\right) \times \left({}_{25}p_{65}^{\text{female}}\right) \\ = \; & 0.339 \times 0.497 = 16.84\% \end{aligned}$$

The above is assuming they are independent events. Are they?

Question: What is the probability that either of them survives from the current age 65 to age 90?

Answer:

$$\begin{aligned} & 1-\left(1-\left({}_{25}p_{65}^{\text{male}}\right)\right) \times \left(1-\left({}_{25}p_{65}^{\text{female}}\right)\right) \\ & 1-(1-0.339) \times (1-0.497) \\ = \; & 66.75\% \end{aligned}$$

Note the probability is almost four times larger. The intuition is that the only event that is excluded is the event in which both people die, which has a $(1-0.339)(1-0.497) = 0.332\,48$ chance. Subtract this from one, and you have the probability that either the male, female or both survive.

Note that we can write:

$$\begin{aligned}&\left({}_tp_x^{\text{male}}\right) \times \left({}_tp_y^{\text{female}}\right) \\ = \;& e^{-\left(\int_x^{x+t}\lambda^{\text{male}}(s)ds+\int_y^{y+t}\lambda^{\text{female}}(s)ds\right)}.\end{aligned} \tag{35}$$

The integral portion can be represented as:

$$-\int_0^t \left(\lambda^{\text{male}}(x+s)+\lambda^{\text{female}}(y+s)\right) ds, \tag{36}$$

which can be combined into one IFM curve:

$$\lambda^{\text{combined}}(s)=\lambda^{\text{male}}(x+s)+\lambda^{\text{female}}(y+s). \tag{37}$$

Finally, if they are both the same age $x=y$ and the parameters for m, b are the same, the combined IFM is simply double the individual IFM.

12 Multiple Decrements

Instead of treating $\lambda(x)$ as a "rate of death" we can think of $\lambda(x)$ as an abstract rate at which individuals are "leaving" a group over time. For example, we might say that:

$$\lambda(x) = \lambda^1(x) + \lambda^2(x) + \lambda^3(x),$$

in which case we have three causes of death, cancer $\lambda^1(x)$, strokes $\lambda^2(x)$ and heart disease $\lambda^3(x)$. We can then model the probability of dying from one cause versus the probability of dying from another.

13 Fitting Discrete Tables to Analytic Laws

What is the best way to "locate" the Gompertz-Makeham or Exponential parameters for the IFM that best "fit" a given mortality table?

- Match the Expected Remaining Lifetime (ERL) or the MRL so that they are the same under both distributions.
- Pick one or two given survival points $({}_tp_x)$ on the mortality table and locate parameters that "fit" this probability.
- Minimizing the distance between the theoretical $f_x(t)$ and the empirical or population $f_x(t)$ over a given range.
- Any combination of the above.

14 Static vs. Dynamics Tables

$$\begin{aligned}({}_5p_{65}^{1940}) &= (1-q_{65}^{1940})\times(1-q_{66}^{1940})\times(1-q_{67}^{1940})\\ &\quad\times(1-q_{68}^{1940})\times(1-q_{69}^{1940})\end{aligned}$$

Alternatively, if we are currently in the year 2005, the IFM could be modeled as:

$$\lambda(z,x) = e^{(1940-z)}\lambda(x) \tag{38}$$

For now all I want to do is alert people to the possibility of a cohort-dependent effect on the law of morality and introduce the notation we will use when these issues are relevant.

$({}_tp_x)$	**Individual Annuity Mortality: Female**		
	1971 Table	1983 Table	1996 Table
55	100%	100%	100%
60	97.6%	98.2%	98.5%
65	93.8%	95.6%	96.2%
70	88.9%	91.4%	92.6%
75	81.2%	84.9%	89.9%
80	68.9%	74.5%	77.5%
85	50.4%	58.6%	62.8%
90	28.1%	37.9%	42.7%
95	10.3%	18.1%	22.1%
100	2.6%	5.9%	8.2%
$E[\mathbf{T}_{55}]$	31.76	34.03	35.22

The above table illustrates this impact over time.

$({}_tp_x)$	**Individual Annuity Mortality: Male**		
	1971 Table	1983 Table	1996 Table
55	100%	100%	100%
60	95.2%	96.6%	97.4%
65	88.6%	91.9%	93.7%
70	79.9%	84.8%	88.0%
75	68.2%	74.2%	79.1%
80	53.0%	59.6%	66.3%
85	35.3%	41.5%	49.6%
90	18.1%	23.4%	31.3%
95	5.6%	10.0%	15.4%
100	0.7%	2.89%	5.55%
$E[\mathbf{T}_{55}]$	27.49	29.70	31.93

So, as you can see, these tables <u>do</u> change over time.

15 Note: Incomplete Gamma function in Excel

Recall that to obtain the Expected Remaining Lifetime (ERL) of the Gomperz-Makeham law of mortality, we were left with the expression:

$$\Gamma(a,c) := \int_c^{\infty} e^{-x} x^{(a-1)} dx, \tag{39}$$

which is the incomplete gamma function. This function is available in Excel, using a slight modification, by using the CDF of the Gamma distribution. We will discuss this distribution in a later chapter/lecture, but for now all we need to now is that:

$$1 - G_a(c) = \int_c^{\infty} \frac{e^{-x} x^{(a-1)}}{\Gamma(a)} dx, \tag{40}$$

where the symbol $G_a(c)$ is the CDF of a Gamma density with parameters c, a.

This leads to:

$$\Gamma(a)\left(1 - G_a(c)\right) = \int_c^{\infty} e^{-x} x^{(a-1)} dx = \Gamma(a,c) \tag{41}$$

The actual syntax in Excel would be:

EXP(GAMMALN(a))*(1-GAMMADIST(c,a,1,TRUE))

Now, in the event that $-1 < a < 0$, we can perform an integration by parts to obtain:

$$\int e^{-x}x^{(a-1)}dx = \frac{1}{a}x^a e^{-x} + \frac{1}{a}\int e^{-x}x^a dx \tag{42}$$

This is synonymous with:

$$\Gamma(a,c) = -\frac{c^a e^{-c}}{a} + \frac{1}{a}\Gamma(a+1,c). \tag{43}$$

In our context, one iteration should be enough to make the implicit gamma parameter positive. The syntax in Excel would then be:

(EXP(GAMMALN($a+1$))*
(1-GAMMADIST(c,$a+1$,1,TRUE)))/a
-(c^a)EXP($-c$)/a

Finally, in the event we need another "round", we use the identity:

$$\begin{aligned}\Gamma(a,c) \;=\; & -\frac{c^a e^{-c}}{a} \\ & +\frac{1}{a}\left(\frac{-c^{(a+1)}e^{-c}}{a+1} + \frac{1}{a+1}\Gamma(a+2,c)\right).\end{aligned} \tag{44}$$

In later chapters I will return to this.