Pension Planning and Investments Under Transaction Costs

M.B. Chiarolla[1] M. Longo[2] G. Stabile[1]

[1]Metodi e modelli per l'economia, il territorio e la finanza,
Università "La Sapienza", Roma

[2]Dipartimento di Discipline Matematiche, Finanza Matematica ed Econometria,
Università Cattolica, Milano

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Outline

- Motivations
  - Saving in a Retirement Plan is a partially irreversible investment
  - Annuity Puzzle
  - Our approach: partially irreversible pension fund contribution
- The model
- Analysis
  - Post-retirement problem
  - Pre-retirement problem
- Conclusion

Saving in a Retirement Plan is a partially irreversible investment

- The contribution to a retirement plan is in general costless also in many countries tax laws encourages individuals to save for retirement
- Early withdrawals from a retirement plan are subject to penalties. This might discourage individuals from investing in retirement plans

The Italian case

- Early withdrawals are allowed in the following cases:
  - ✓ medical expenses: maximal amount 75%, tax penalty of 10%
  - ✓ buying a house: maximal amount 75%, tax penalty of 23%
  - ✓ other motives: maximal amount 30%, tax penalty of 23%

- At the end of 2010, the membership to voluntary pension plans was only 23% (Covip, Commissione di Vigilanza su i Fondi Pensione)

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Annuity Puzzle

- Yaari showed that individuals with no bequest motive annuitize all their wealth at retirement (Yaari, RES 1965)
- "empirical evidence shows that very few people purchase life annuities" (Brown, J.Publ.Ec. 2001)

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Literature on the puzzle's explanation

- *Bequest* (Hurd, Econometrica 1989)
- *Adverse selection* (Abel, Econometrica 1986)
- *Annuity costs* (Friedman and Warshawsky, QJE 1990)
- *Irreversible annuitization* (Milevsky and Young, JEDC 2007)

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Our approach: partially irreversible pension fund contribution

- Illiquidity during the accumulation of the retirement account
- Transaction costs machinery (Magill and Constantinides, 1976; Davis and Norman, 1990; Shreve and Soner, 1994)

The model

Consider an individual that maximizes the expected discounted utility from consumptions over his random lifetime horizon $[0, \tau]$
Let T be the deterministic retirement time

- **Pre-retirement**: on $[0, T \wedge \tau)$
  The individual may invest in the financial market while contributing to a voluntary pension account. At times, to finance consumption, he might withdraw funds from the pension account, incurring in penalties
- **Retirement**: at T
  The pension account is converted into a constant riskless life annuity
- **Post-retirement**: on $[T \wedge \tau, \tau]$
  The individual receives the annuity income while still investing in the financial market

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The Market

On a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\right)$

- **Uncertainty:** 2-dim Brownian Motion $(Z_1(t), Z_2(t))$
- **The frictionless Financial market:**

$$\begin{cases} dB(t) = rB(t)\,dt \\ dA(t) = A(t)\left[\mu_a dt + \sigma_a dZ_1(t)\right] \end{cases}$$

- **The Pension Fund Account:**

$$dP(t) = P(t)\left[\mu_p dt + \sigma_p dZ_1(t)\right]$$

We assume that $\mu_p \geq \mu_a$ and $\sigma_p \leq \sigma_a$

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The individual wealth

- **Our state variables are**
  - ✓ $X(\cdot)$ liquid wealth (e.g. allocated in the Financial Market)
  - ✓ $Y(\cdot)$ illiquid wealth (i.e. the wealth invested in the pension fund)

- **Income flow:**

$$dI(t) = X(t)\left[\mu_I dt + \sigma_I dZ(t)\right],$$

$Z(t) = \rho Z_1(t) + \sqrt{1-\rho^2} Z_2(t),\ \rho \in (-1, 1)$
(unhedgeable income, Hipp and Plum, F&S 2003)

- **Market price of a continuous monetary unit Life Annuity**

$$a_{T,r} = \int_0^\infty e^{-rs} {}_s p_T^{ob} ds$$

${}_s p_T^{ob}$ is the survivor probability used by the insurer

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Control variables

- ✓ $c(\cdot)$ consumption rate
- ✓ $\pi(\cdot)$ amount of liquid wealth invested in stocks
- ✓ $M(\cdot)$ cumulative withdrawal from the pension fund
- ✓ $L(\cdot)$ cumulative contribution to the pension fund

Early withdrawals are subject to a **cost proportional to the amount**, i.e. if the individual withdraws one monetary unit from the pension account, he actually receives $(1-\lambda)$ $\quad (\lambda \in (0,1))$

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State variables dynamics

- *Liquid wealth dynamics*
  - $$dX(t) = [rX(t) + (\mu_a - r)\pi(t) - c(t)]\,dt + \sigma_a\pi(t)\,dZ_1(t) + dI(t) - dL(t) + (1-\lambda)\,dM(t), \qquad \text{if } 0 \le t < T$$
  - $$dX(t) = [rX(t) + (\mu_a - r)\pi(t) - c(t)]\,dt + \sigma_a\pi(t)\,dZ_1(t) + \frac{Y(T)}{a_{T,r}}dt, \qquad \text{if } t \ge T$$
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Agent's profile

- Preferences:

$$u(c) = \frac{c^\gamma}{\gamma}, \qquad \gamma < 1,\ \gamma \neq 0$$

- Agent random lifetime:

$$\tau : \Omega \to (0, +\infty)$$

- The survival probability:

$${}_s p_t := \mathbb{P}(\tau \geq t + s \mid \tau \geq t) = e^{-\int_t^{t+s} \theta(z) dz}$$

with θ increasing and continuous.

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The problem

Maximize

$$J(t,x,y,\xi) := \mathbb{E}\left[\int_t^{\tau} e^{-\delta(s-t)} u(c(s))\, ds\right], \qquad (1)$$

over all admissible controls $\xi := (c, \pi, L, M)$

Assuming independency between financial and demographic uncertainty, (1) may be written as

$$J(t,x,y;\xi) = \mathbb{E}\left[\int_t^{\infty} e^{-\int_t^s (\delta+\theta(z))dz} u(c(s))\, ds\right]$$

The indirect utility (or **value function**) is

$$v(t,x,y) := \sup_{\xi} J(t,x,y,\xi)$$

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Problem reduction

Markovian structure

+

Dynamic Programming Principle

$\Downarrow$

$$v(t,x,y) = \sup_{\xi} \mathbb{E}\Big[\int_t^T e^{-\int_t^u (\delta+\theta(z))dz} u(c(u))du + e^{-\int_t^T (\delta+\theta(z))dz} v^{post}(T, X^{\xi}_{t,x,y}(T), Y^{\xi}_{t,x,y}(T))\Big]$$

The problem splits into:

1. the Post-retirement problem ($t \geq T$)
2. the Pre-retirement problem ($t < T$)

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Post-retirement problem: $t \geq T$

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subject to

$$dX(s) = [rX(s) + (\mu_a - r)\pi(s) + \frac{y}{a_{T,r}} - c(s)]ds + \sigma_a \pi(s) dZ_1(s)$$

Merton type consumption/investment problem:

- ✓ deterministic mortality force
- ✓ constant income from annuity
- ✓ infinite horizon

Post-retirement problem: $t \geq T$

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The Guess and Verify method:

- ✓ HJB for the value function
- ✓ HJB analysis (explicit solution)
- ✓ Optimal controls and Verification Theorem

$$\max_{(c,\pi)\in[0,\infty)\times\mathbb{R}} \left\{ v_t + \mathcal{L}^{c,\pi}[v] + u(c) - (\delta + \theta(t))v \right\} = 0, \qquad (2)$$

where $\mathcal{L}^{c,\pi}$ is the differential operator associated to X

$$\mathcal{L}^{c,\pi}[w](t,x,y) = \frac{1}{2}\sigma_a^2\pi^2 w_{xx} + rxw_x + \pi(\mu_a - r)w_x + \frac{y}{a_{T,r}}w_x - cw_x,$$

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The Post-retirement value function

$$v^{post}(t,x,y) = \frac{(k(t))^{1-\gamma}}{\gamma}\left(x + \frac{y}{r\, a_{T,r}}\right)^{\gamma}$$

where

$$k(t) = \int_t^{\infty} \exp\left\{-\frac{1}{1-\gamma}\int_t^u \left[\frac{1}{2}\left(\frac{\mu_a - r}{\sigma_a}\right)^2 \frac{\gamma}{\gamma-1} - r\gamma + \delta + \theta(s)\right] ds\right\} du$$

The Post-retirement optimal policies

$$c^*(t,x,y) = \frac{1}{k(t)}\left(x + \frac{y}{r\, a_{T,r}}\right)$$

$$\pi^*(t,x,y) = \frac{(\mu_a - r)}{\sigma_a^2 (1-\gamma)}\left(x + \frac{y}{r\, a_{T,r}}\right)$$

- strategies take into account the future pension stream
- the consumption rate increases in time, all other things the same
- if the mortality force goes to ∞ then $c^* \to \infty$
- π^* is not affected by demographic uncertainty

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Pre-retirement problem $t \in [0, T)$

$$v(t,x,y) = \sup_{\xi} \mathbb{E}\Big[\int_t^T e^{-\int_t^u (\delta+\theta(z))dz} u(c(u))du + e^{-\int_t^T (\delta+\theta(z))dz} v^{post}(T, X^{\xi}_{t,x,y}(T), Y^{\xi}_{t,x,y}(T)) \Big]$$

- **Consumption/investment problem with:**
  - ✓ proportional transaction costs
  - ✓ demographic uncertainty
  - ✓ stochastic income
  - ✓ finite time horizon

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\begin{aligned}
v(t,x,y) &= \sup_{\xi} \mathbb{E}\Big[\int_t^T e^{-\int_t^u (\delta+\theta(z))dz} u(c(u))du \\
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The Solvency Region

$$\begin{cases} dX(t) & = & [rX(t) + (\mu_a - r)\pi(t) - c(t)]dt + \sigma_a \pi(t) dZ_1(t) \\ & & +dI(t) - dL(t) + (1-\lambda)dM(t), \\ dY(t) & = & Y(t)\left[\mu_p dt + \sigma_p dZ_1(t)\right] + dL(t) - dM(t) \end{cases}$$

Let the individual hold a position (x, y), then he gets

- $x + (1-\lambda)y$ if he liquidates instantaneously the holding in the pension account
- $x + y$ if he transfers all liquid wealth into the pension account

Assume the individual cannot borrow against the pension account, i.e. $y \geq 0$, then

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Heuristic and HJB equation (1)

Fix $t \in [0, T)$ and $(x, y) \in \mathcal{S}_T$

- If the individual does *not transfer money to or from the fund*, then

$$\sup_{(c,\pi)\in\mathbb{R}^+\times\mathbb{R}} \left\{ v_t + \mathcal{M}^{c,\pi}[v] + u(c) - (\delta + \theta(t))v \right\} \geq 0 \qquad (3)$$

where

$$\mathcal{M}^{c,\pi}[g] = \frac{1}{2}\left[\left(\sigma_a \pi + \rho\sigma_I x\right)^2 + (1-\rho^2)\sigma_I^2 x^2 \right] g_{xx} + \frac{1}{2} y^2 \sigma_p^2 g_{yy}$$
$$+ \left(\sigma_a \pi + \rho\sigma_I x\right)\sigma_p y g_{xy} + \left[\pi(\mu_a - r) + rx + \mu_I x - c \right] g_x + \mu_p y g_y .$$

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Heuristic and HJB equation (2)

- If the individual *withdraws immediately* $dM = \varepsilon > 0$ *from the fund and then proceeds optimally*, we have

$$v(t,x,y) \geq v(t, x + (1-\lambda)\varepsilon, y - \varepsilon)$$

Dividing by ε and letting $\varepsilon \to 0$, we obtain

$$-(1-\lambda)v_x + v_y \geq 0. \tag{4}$$

- Similarly, the policy *contribute* $dL = \varepsilon$ *and then proceed optimally*, yields the condition

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Pre-retirement HJB formulation

We expect the value function to satisfy in some sense the following variational inequality:

$$\min\left\{ \sup_{c,\pi}\{v_t+\mathcal{M}^{c,\pi}[v]+u(c)-(\delta+\theta(t))v\}; -(1-\lambda)v_x+v_y; v_x-v_y \right\} = 0$$

with boundary condition

$$v(T,x,y) = v^{post}(T,x,y)$$

Lipschitz continuous controls (1)

Let L and M such that

$$L(t) = \int_0^t l(s)\, ds, \;\; M(t) = \int_0^t m(s)\, ds, \;\; 0 \leq l(s), m(s) \leq k$$

Then, the state dynamics become

$$\begin{cases} dX(t) &= [rX(t) + (\mu_a - r)\pi(t) - c(t) - l(t) + (1-\lambda)m(t)]dt + \\ & \quad \sigma_a \pi(t) dZ_1(t) + dI(t), \\ dY(t) &= Y(t)\left[\mu_p dt + \sigma_p dZ_1(t)\right] + l(t)\, dt - m(t)\, dt \end{cases}$$

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The HJB equation becomes

$$\max_{0\le l,m\le k}\left\{v_t+\mathcal{M}^{c,\pi}[v]+(v_y-v_x)l+((1-\lambda)v_x-v_y)m+u(c)-(\delta+\theta(t))v\right\}=0$$

The maximizers are

$$l=\begin{cases} k & if \quad v_y\ge v_x \\ 0 & if \quad v_y<v_x \end{cases}\quad,\quad m=\begin{cases} k & if \quad v_y\le(1-\lambda)v_x \\ 0 & if \quad v_y>(1-\lambda)v_x \end{cases}$$

Then, the optimal contribution/withdrawal policies are **bang-bang**

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For $t \in [0, T)$,

- the value function is
  - ✓ concave in (x, y)
  - ✓ $v(t, \rho x, \rho y) = \rho^{\gamma} v(t, x, y), \qquad \rho > 0,$

  where $(x, y) \in \mathcal{S}_T$

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  - ✓ *Withdrawal*
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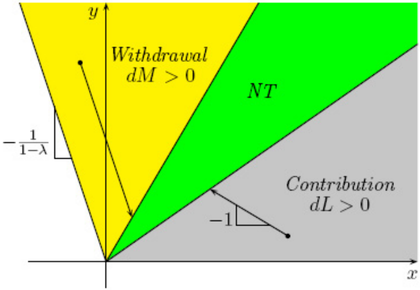

The lines delimiting the *NT* region depend on time and on the parameters of the model

The Pre-retirement optimal policies

$$c^*(t) = (v_x)^{\frac{1}{\gamma-1}}$$

$$\pi^*(t) = -\underbrace{\frac{(\mu_a - r)\, v_x}{\sigma_a^2 v_{xx}}}_{\text{Merton}} - \underbrace{\frac{\sigma_p Y^*(t)\, v_{xy}}{\sigma_a v_{xx}}}_{\text{fund}} - \underbrace{\frac{\rho \sigma_I X^*(t)}{\sigma_a}}_{\text{income}}$$

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State space reduction

The homothetic property of the value function allows for the following state space reduction.
Define

$$w(t,z) := v(t,z,1), \qquad z \in (\lambda - 1, +\infty).$$

then

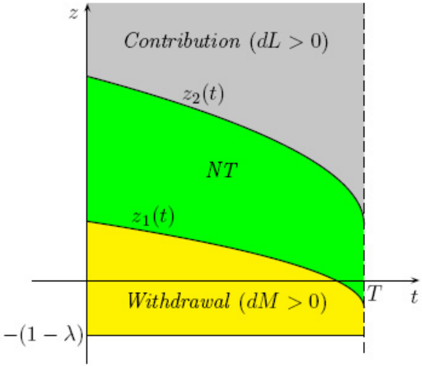
$$v(t,x,y) = y^{\gamma} w\left(t, \frac{x}{y}\right),$$

where w solves the following variational inequality:

$$\min\left\{ w_t + \beta_1 z w_z - \beta_2 \frac{w_z^2}{w_{zz}} + \beta_3 \left(w_z\right)^{\frac{\gamma}{\gamma-1}} + \alpha\left(t\right) w; \right.$$
$$\left. -\left(z+1-\lambda\right) w_z + \gamma w; \quad \left(z+1\right) w_z - \gamma w \right\} = 0, \tag{6}$$

for every $(t,z) \in [0,T) \times (\lambda - 1, +\infty)$, with $\beta_1, \beta_2, \beta_3$ real constants dependent on parameters and $\alpha(t)$ a deterministic function

The *Withdrawal*, *NT*, and *Contribution* regions in the reduced state space:



The curves delimiting the *NT* region depend on the parameters of the model

Solutions in the transaction regions

A family of solutions of the reduced HJB in the *Withdrawal* region is:

$$w(t,z) = A(t)\Big(z + (1-\lambda)\Big)^{\gamma}, \quad \forall (t,z) \in (0,T) \times (\lambda - 1, z_1(t)),$$

whereas, in the *Contribution* region, a class of solutions of the HJB is:

$$w(t,z) = B(t)\Big(z + 1\Big)^{\gamma}, \quad \forall (t,z) \in (0,T) \times (z_2(t), +\infty).$$

The functions $A(t), B(t), z_1(t), z_2(t)$ will be determined endogenously by means of the so called *principle of smooth-fit*, once we have a reasonable family of solutions in the *NT* region

Final remarks

- So far, we have splitted the individual's problem into two separated problems concerning:
  - ✓ the post-retirement period (explicit solutions)
  - ✓ the pre-retirement period (still in progress)
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