Pension Planning and Investments
Under Transaction Costs

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Outline

Motivations
-Saving in a Retirement Plan is a partially irreversible investment
-Annuity Puzzle
-Our approach: partially irreversible pension fund contribution

The model

Analysis
-Post-retirement problem
-Pre-retirement problem

Conclusion
Savings in a Retirement Plan is a partially irreversible investment

- The contribution to a retirement plan is in general costless also in many countries tax laws encourages individuals to save for retirement.

- Early withdrawals from a retirement plan are subject to penalties. This might discourage individuals from investing in retirement plans.
The Italian case

- Early withdrawals are allowed in the following cases:
  - medical expenses: maximal amount 75%, tax penalty of 10%
  - buying a house: maximal amount 75%, tax penalty of 23%
  - other motives: maximal amount 30%, tax penalty of 23%

- At the end of 2010, the membership to voluntary pension plans was only 23% (Covip, Commissione di Vigilanza su i Fondi Pensione)
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Annuity Puzzle

- Yaari showed that individuals with no bequest motive annuitize all their wealth at retirement (Yaari, RES 1965)

- “empirical evidence shows that very few people purchase life annuities” (Brown, J.Publ.Ec. 2001)
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Literature on the puzzle’s explanation

- *Bequest* (Hurd, Econometrica 1989)
- *Adverse selection* (Abel, Econometrica 1986)
- *Annuity costs* (Friedman and Warshawsky, QJE 1990)
- *Irreversible annuitization* (Milevsky and Young, JEDC 2007)
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Our approach: partially irreversible pension fund contribution

- Illiquidity during the accumulation of the retirement account

- Transaction costs machinery (Magill and Constantinides, 1976; Davis and Norman, 1990; Shreve and Soner, 1994)
The model

Consider an individual that maximizes the expected discounted utility from consumptions over his random lifetime horizon \([0, \tau]\).

Let \(T\) be the deterministic retirement time.

- **Pre-retirement**: on \([0, T \wedge \tau)\)
  The individual may invest in the financial market while contributing to a voluntary pension account. At times, to finance consumption, he might withdraw funds from the pension account, incurring in penalties.

- **Retirement**: at \(T\)
  The pension account is converted into a constant riskless life annuity.

- **Post-retirement**: on \([T \wedge \tau, \tau]\)
  The individual receives the annuity income while still investing in the financial market.
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The Market

On a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\)

- **Uncertainty:** 2-dim Brownian Motion \((Z_1(t), Z_2(t))\)

- The frictionless Financial market:

\[
\begin{aligned}
\begin{cases}
    dB(t) = rB(t) \, dt \\
    dA(t) = A(t) [\mu_a dt + \sigma_a dZ_1(t)]
\end{cases}
\end{aligned}
\]

- The Pension Fund Account:

\[
dP(t) = P(t) [\mu_p dt + \sigma_p dZ_1(t)]
\]

We assume that \(\mu_p \geq \mu_a\) and \(\sigma_p \leq \sigma_a\)
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The individual wealth

- **Our state variables are**
  - $X(\cdot)$ liquid wealth (e.g. allocated in the Financial Market)
  - $Y(\cdot)$ illiquid wealth (i.e. the wealth invested in the pension fund)

- **Income flow:**
  \[
  dl(t) = X(t) \left[ \mu_1 dt + \sigma_1 dZ(t) \right],
  \]
  \[
  Z(t) = \rho Z_1(t) + \sqrt{1 - \rho^2} Z_2(t), \quad \rho \in (-1, 1)
  \]
  (unhedgeable income, Hipp and Plum, F&S 2003)

- **Market price of a continuous monetary unit Life Annuity**
  \[
  a_{T,r} = \int_0^\infty e^{-rs} s p^o_T ds
  \]
  $s p^o_T$ is the survivor probability used by the insurer
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Control variables

✓ $c(\cdot)$ consumption rate
✓ $\pi(\cdot)$ amount of liquid wealth invested in stocks
✓ $M(\cdot)$ cumulative withdrawal from the pension fund
✓ $L(\cdot)$ cumulative contribution to the pension fund

Early withdrawals are subject to a cost proportional to the amount, i.e. if the individual withdraws one monetary unit from the pension account, he actually receives $(1 - \lambda)$ ($\lambda \in (0, 1)$)
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State variables dynamics

- **Liquid wealth dynamics**

\[ dX(t) = [rX(t) + (\mu_a - r)\pi(t) - c(t)] \, dt + \sigma_a\pi(t) \, dZ_1(t) \]
\[ + \, dl(t) - dL(t) + (1 - \lambda) \, dM(t), \quad \text{if } 0 \leq t < T \]
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\[ + \, \frac{Y(T)}{aT, r} \, dt, \quad \text{if } t \geq T \]

- **Illiquid wealth dynamics**

\[ dY(t) = Y(t) [\mu_p \, dt + \sigma_p \, dZ_1(t)] + dL(t) - dM(t), \quad \text{if } 0 \leq t \leq T \]
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Agent’s profile

▶ Preferences:

\[ u(c) = \frac{c^\gamma}{\gamma}, \quad \gamma < 1, \gamma \neq 0 \]

▶ Agent random lifetime:

\[ \tau : \Omega \rightarrow (0, +\infty) \]

▶ The survival probability:

\[ s_p(t) := \mathbb{P}(\tau \geq t + s \mid \tau \geq t) = e^{-\int_t^{t+s} \theta(z) dz} \]

with \( \theta \) increasing and continuous.
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The problem

Maximize

\[ J(t, x, y, \xi) := \mathbb{E} \left[ \int_t^T e^{-\delta(s-t)} u(c(s)) \, ds \right], \quad (1) \]

over all admissible controls \( \xi := (c, \pi, L, M) \)

Assuming independency between financial and demographic uncertainty, (1) may be written as

\[ J(t, x, y; \xi) = \mathbb{E} \left[ \int_t^\infty e^{-\int_t^s (\delta + \theta(z)) \, dz} u(c(s)) \, ds \right] \]

The indirect utility (or value function) is

\[ v(t, x, y) := \sup_{\xi} J(t, x, y, \xi) \]
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Problem reduction

Markovian structure

\[ v(t, x, y) = \sup_{\xi} \mathbb{E} \left[ \int_t^T e^{-\int_t^u (\delta + \theta(z))dz} u(c(u))du \right. \]

\[ + e^{-\int_t^T (\delta + \theta(z))dz} v^{post}(T, X_{t,x,y}(T), Y_{t,x,y}(T)) \]

The problem splits into:

1. the Post-retirement problem \((t \geq T)\)
2. the Pre-retirement problem \((t < T)\)
Problem reduction

Markovian structure

Dynamic Programming Principle

\[ v(t, x, y) = \sup_{\xi} \mathbb{E} \left[ \int_{t}^{T} e^{-\int_{t}^{u} (\delta + \theta(z)) \, dz} u(c(u)) \, du \right. \]

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1. the Post-retirement problem \( t \geq T \)

2. the Pre-retirement problem \( t < T \)
Post-retirement problem: $t \geq T$

$$v(t, x, y) = \sup_{(c, \pi)} \mathbb{E} \left[ \int_t^\infty e^{-\int_t^s (\delta + \theta(z)) \, dz} u(c(s)) \, ds \right],$$

subject to

$$dX(s) = \left[ rX(s) + (\mu_a - r)\pi(s) + \frac{y}{a_T, r} - c(s) \right] ds + \sigma_a \pi(s) dZ_1(s)$$

Merton type consumption/investment problem:

- deterministic mortality force
- constant income from annuity
- infinite horizon
Post-retirement problem: $t \geq T$

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Merton type consumption/investment problem:

✓ deterministic mortality force
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✓ infinite horizon
The Guess and Verify method:

✓ HJB for the value function

✓ HJB analysis (explicit solution)

✓ Optimal controls and Verification Theorem

\[
\max_{(c, \pi) \in [0, \infty) \times \mathbb{R}} \left\{ v_t + \mathcal{L}^{c,\pi}[v] + u(c) - (\delta + \theta(t))v \right\} = 0,
\]

where \( \mathcal{L}^{c,\pi} \) is the differential operator associated to \( X \)

\[
\mathcal{L}^{c,\pi}[w](t, x, y) = \frac{1}{2} \sigma^2_a \pi^2 w_{xx} + rxw_x + \pi(\mu_a - r)w_x + \frac{y}{a_{T,r}} w_x - cw_x,
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with \( w \in C^{1,2}([T, \infty) \times \mathbb{R}^2) \).
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The Guess and Verify method:

✓ HJB for the value function
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$$\max_{(c,\pi) \in [0,\infty) \times \mathbb{R}} \left\{ v_t + L_{c,\pi}^c [v] + u(c) - (\delta + \theta(t))v \right\} = 0,$$

where $L_{c,\pi}^c$ is the differential operator associated to $X$

$$L_{c,\pi}^c [w](t, x, y) = \frac{1}{2} \sigma^2 a^2 \pi^2 w_{xx} + rxw_x + \pi (\mu a - r)w_x + \frac{y}{a_{T,r}} w_x - cw_x,$$

with $w \in C^{1,2}([T, \infty) \times \mathbb{R}^2)$. 
The Post-retirement value function

\[ v_{\text{post}}(t, x, y) = \frac{(k(t))^{1-\gamma}}{\gamma} \left( x + \frac{y}{r \, a_T, r} \right)^\gamma \]

where

\[ k(t) = \int_t^\infty \exp \left\{ -\frac{1}{1-\gamma} \int_t^\infty \left[ \frac{1}{2} \left( \frac{\mu a - r}{\sigma a} \right)^2 \frac{\gamma}{\gamma - 1} - r\gamma + \delta + \theta(s) \right] \, ds \right\} \, du \]
The Post-retirement optimal policies

\[
c^*(t, x, y) = \frac{1}{k(t)} \left( x + \frac{y}{r a_{T,r}} \right)
\]

\[
\pi^* (t, x, y) = \frac{(\mu a - r)}{\sigma_a^2 (1 - \gamma)} \left( x + \frac{y}{r a_{T,r}} \right)
\]

- strategies take into account the future pension stream
- the consumption rate increases in time, all other things the same
- if the mortality force goes to \( \infty \) then \( c^* \to \infty \)
- \( \pi^* \) is not affected by demographic uncertainty
The Post-retirement optimal policies

\[ c^* (t, x, y) = \frac{1}{k(t)} \left( x + \frac{y}{r a_{T,r}} \right) \]

\[ \pi^* (t, x, y) = \frac{(\mu_a - r)}{\sigma^2_a (1 - \gamma)} \left( x + \frac{y}{r a_{T,r}} \right) \]

- strategies take into account the future pension stream
- the consumption rate increases in time, all other things the same
- if the mortality force goes to \( \infty \) then \( c^* \rightarrow \infty \)
- \( \pi^* \) is not affected by demographic uncertainty
Pre-retirement problem $t \in [0, T)$

\[
\nu(t, x, y) = \sup_{\xi} \mathbb{E} \left[ \int_t^T e^{-\int_t^u (\delta + \theta(z)) dz} u(c(u)) du + e^{-\int_t^T (\delta + \theta(z)) dz} \nu^{post}(T, X_{t,x,y}(T), Y_{t,x,y}(T)) \right]
\]

- Consumption/investment problem with:
  - proportional transaction costs
  - demographic uncertainty
  - stochastic income
  - finite time horizon
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- Consumption/investment problem with:
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The Solvency Region

\[
\begin{align*}
    dX(t) &= [rX(t) + (\mu_a - r)\pi(t) - c(t)]dt + \sigma_a\pi(t)dZ_1(t) \\
    &\quad + dl(t) - dL(t) + (1 - \lambda)dM(t), \\
    dY(t) &= Y(t)[\mu_p dt + \sigma_p dZ_1(t)] + dL(t) - dM(t)
\end{align*}
\]

Let the individual hold a position \((x, y)\), then he gets

- \(x + (1 - \lambda)y\) if he liquidates instantaneously the holding in the pension account
- \(x + y\) if he transfers all liquid wealth into the pension account

Assume the individual cannot borrow against the pension account, i.e. \(y \geq 0\), then

- the Solvency region is

\[
S_T = \left\{(x, y) : x + (1 - \lambda)y \geq 0, \ y \geq 0\right\}
\]
The Solvency Region

$$\begin{align*}
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The Solvency Region

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\]
Heuristic and HJB equation (1)

Fix \( t \in [0, T) \) and \((x, y) \in S_T\)

- If the individual does not transfer money to or from the fund, then

\[
\sup_{(c, \pi) \in \mathbb{R}^+ \times \mathbb{R}} \left\{ v_t + M^{c, \pi}[v] + u(c) - (\delta + \theta(t)) v \right\} \geq 0 \tag{3}
\]

where

\[
M^{c, \pi}[g] = \frac{1}{2} \left[ (\sigma_a \pi + \rho \sigma_1 x)^2 + (1 - \rho^2) \sigma_1^2 x^2 \right] g_{xx} + \frac{1}{2} y^2 \sigma_p^2 g_{yy} \\
+ (\sigma_a \pi + \rho \sigma_1 x) \sigma_p y g_{xy} + \left[ \pi (\mu_a - r) + rx + \mu_1 x - c \right] g_x + \mu_p y g_y.
\]
Heuristic and HJB equation (1)

Fix $t \in [0, T)$ and $(x, y) \in S_T$

- If the individual does not transfer money to or from the fund, then

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where

$$M^{c, \pi}[g] = \frac{1}{2} \left[ (\sigma_{a\pi} + \rho\sigma_{Ix})^2 + (1 - \rho^2)\sigma_{ix}^2 \right] g_{xx} + \frac{1}{2} y^2 \sigma_{py}^2 g_{yy}$$

$$+ (\sigma_{a\pi} + \rho\sigma_{Ix}) \sigma_{py} g_{xy} + \left[ \pi (\mu_a - r) + rx + \mu_{Ix}x - c \right] g_x + \mu_{py} g_y.$$
Heuristic and HJB equation (2)

- If the individual withdraws immediately $dM = \varepsilon > 0$ from the fund and then proceeds optimally, we have

$$v(t, x, y) \geq v(t, x + (1 - \lambda)\varepsilon, y - \varepsilon)$$

Dividing by $\varepsilon$ and letting $\varepsilon \to 0$, we obtain

$$-(1 - \lambda)v_x + v_y \geq 0.$$  \hspace{1cm} (4)

- Similarly, the policy contribute $dL = \varepsilon$ and then proceed optimally, yields the condition

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$$v_x - v_y \geq 0.$$  \hspace{1cm} (5)
Pre-retirement HJB formulation

We expect the value function to satisfy in some sense the following variational inequality:

$$\min \left\{ \sup_{c, \pi} \left\{ v_t + M^{c, \pi}[v] + u(c) - (\delta + \theta(t))v \right\}; - (1 - \lambda)v_x + v_y; v_x - v_y \right\} = 0$$

with boundary condition

$$v(T, x, y) = v^{post}(T, x, y)$$
Lipschitz continuous controls (1)

Let $L$ and $M$ such that

$$L(t) = \int_0^t l(s) \, ds, \quad M(t) = \int_0^t m(s) \, ds, \quad 0 \leq l(s), m(s) \leq k$$

Then, the state dynamics become

$$
\begin{align*}
\frac{dX(t)}{dt} &= [rX(t) + (\mu_a - r)\pi(t) - c(t) - l(t) + (1 - \lambda)m(t)]dt + \\
& \quad \sigma_a\pi(t)dZ_1(t) + dl(t), \\
\frac{dY(t)}{dt} &= Y(t)[\mu_p dt + \sigma_p dZ_1(t)] + l(t)dt - m(t)dt
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\end{align*}
\]
Lipschitz continuous controls (2)

The HJB equation becomes

$$\max_{0 \leq l, m \leq k} \left\{ v_t + \mathcal{L}_{c, \pi}^c[v] + (v_y - v_x)l + ((1 - \lambda)v_x - v_y)m + u(c) - (\delta + \theta(t))v \right\} = 0$$

The maximizers are

$$l = \begin{cases} k & \text{if } v_y \geq v_x \\ 0 & \text{if } v_y < v_x \end{cases}, \quad m = \begin{cases} k & \text{if } v_y \leq (1 - \lambda)v_x \\ 0 & \text{if } v_y > (1 - \lambda)v_x \end{cases}$$

Then, the optimal contribution/withdrawal policies are **bang-bang**
Lipschitz continuous controls (2)

The HJB equation becomes

$$\max_{0 \leq l, m \leq k} \left\{ v_t + \mathcal{M}^c, \pi [v] + (v_y - v_x) l + ((1 - \lambda) v_x - v_y) m + u(c) - (\delta + \theta(t)) v \right\} = 0$$

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Then, the optimal contribution/withdrawal policies are **bang-bang**
For $t \in [0, T)$,

- the value function is
  - concave in $(x, y)$
  - $v(t, \rho x, \rho y) = \rho^\gamma v(t, x, y)$, $\rho > 0$,

where $(x, y) \in S_T$

- the Solvency region splits into
  - Withdrawal
  - No Transaction (NT)
  - Contribution

The three regions are connected sets separated by straight lines.
For $t \in [0, T)$,

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The three regions are connected sets separated by straight lines
The lines delimiting the $NT$ region depend on time and on the parameters of the model.
The Pre-retirement optimal policies

\[ c^*(t) = (v_x)^{\frac{1}{\gamma - 1}} \]

\[ \pi^*(t) = -\frac{(\mu_a - r) v_x}{\sigma_a^2 v_{xx}} - \frac{\sigma_p Y^*(t) v_{xy}}{\sigma_a v_{xx}} - \frac{\rho \sigma_I X^*(t)}{\sigma_a} \]

\[ (L^*(t), M^*(t)) = \begin{cases} \text{local times that keep} \\ (X^*(t), Y^*(t)) \text{ in } NT \end{cases} \]

- \( c^* \) has the usual Merton form in terms of \( v_x \)
- income uncertainty \( \Rightarrow \) prudent financial investment
- ambiguous effect of the pension fund
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\end{array} \right. \]

- \( c^* \) has the usual Merton form in terms of \( v_x \)
- income uncertainty \( \Rightarrow \) prudent financial investment
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State space reduction

The homothetic property of the value function allows for the following state space reduction.

Define

\[ w(t, z) := v(t, z, 1), \quad z \in (\lambda - 1, +\infty). \]

then

\[ v(t, x, y) = y^\gamma w \left( t, \frac{x}{y} \right), \]

where \( w \) solves the following variational inequality:

\[
\min \left\{ w_t + \beta_1 zw_z - \beta_2 \frac{w_z^2}{w_{zz}} + \beta_3 (w_z)^{\gamma - 1} + \alpha(t) w; \right.

\left. - (z + 1 - \lambda) w_z + \gamma w; \quad (z + 1) w_z - \gamma w \right\} = 0, \tag{6}
\]

for every \((t, z) \in [0, T) \times (\lambda - 1, +\infty)\), with \( \beta_1, \beta_2, \beta_3 \) real constants dependent on parameters and \( \alpha(t) \) a deterministic function.
The *Withdrawal, NT, and Contribution* regions in the reduced state space:

The curves delimiting the $NT$ region depend on the parameters of the model.
Solutions in the transaction regions

A family of solutions of the reduced HJB in the *Withdrawal* region is:

$$w(t, z) = A(t) \left( z + (1 - \lambda) \right)^{\gamma}, \quad \forall (t, z) \in (0, T) \times (\lambda - 1, z_1(t)),$$

whereas, in the *Contribution* region, a class of solutions of the HJB is:

$$w(t, z) = B(t) \left( z + 1 \right)^{\gamma}, \quad \forall (t, z) \in (0, T) \times (z_2(t), +\infty).$$

The functions $A(t), B(t), z_1(t), z_2(t)$ will be determined endogenously by means of the so called *principle of smooth-fit*, once we have a reasonable family of solutions in the *NT* region.
Final remarks

- So far, we have split the individual’s problem into two separated problems concerning:
  - the post-retirement period (explicit solutions)
  - the pre-retirement period (still in progress)

- Further analysis is needed in order to answer the original question

- The boundary of $NT$ and its dependence on the problem’s parameters

- Numerical analysis
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